Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Production of Scalar and Pseudoscalar Mesons by Photons

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(Received May 15, 1950)

Introduction

The cross section for the production of a scalar meson by nucleon absorbing a photon was first calculated by Nordheim and Nordheim,[1] and Yukawa and his co-workers.[2] They obtained merely rough results. An exact expression of the differential cross section for the production of positive scalar and pseudoscalar mesons by the same process was first obtained by the present author,[3] but the result for the total cross section was still approximate. The later and recently published theoretical results were still either approximate or qualitative.[4][5][6][7][8]

The recent advance of the laboratory and cosmic-ray experiments requires the detailed conclusion of theory. Therefore the present paper is, as a supplement to the previous paper,[3] intended to give the theoretical conclusion in detail to the differential and total cross sections for the process $\gamma+P\rightarrow N+\pi^+$, $\gamma+N\rightarrow P+\pi^-$, and $\gamma+P\rightarrow P+\pi^0$ according to the scalar and pseudoscalar meson theories.

The present calculation is based on the customary perturbation method partly using the method of unitary transformation and $(eg)^2$ terms only are taken into account. The pure perturbation method of A[3] gives the same result. A calculation according to the modern covariant formalism of Tomonaga and Schwinger has recently been begun to include higher order terms by Koba, Kotani end Nakai[8]though the final result has not yet been obtained.

§ 1. Interaction Energy

The Lagrangian function of a system which consists of nucleons, charged scalar or pseudoscalar mesons, and photons is given by

$$L=(4\pi)^{-1}(E_0{}^*E_0-\boldsymbol{E}^*\boldsymbol{E}-\mu^2U^*U)-(QU^*+Q^*U)$$
$$+\mu^{-1}(\boldsymbol{P}\boldsymbol{E}^*+\boldsymbol{P}^*\boldsymbol{E}-P_0E_0{}^*-P_0{}^*E_0)+L_{\text{nucleon+photon}} \tag{1.1}$$

where

$$\boldsymbol{E}=-grad\,U+ie\boldsymbol{A}U,\qquad E_0=\partial U/\partial t+ieA_0U \tag{1.2}$$

$$Q=g_1\psi^\dagger\tau\rho_3\psi,\ \boldsymbol{P}=g_2\psi^\dagger\tau\rho_1\sigma\psi,\ P_0=g_2\psi^\dagger\tau\psi \quad \text{for scalar mesons} \tag{1.3a}$$

$$Q=f_1\psi^\dagger\tau\rho_2\psi,\ \boldsymbol{P}=f_2\psi^\dagger\tau\sigma\psi,\ P_0=f_2\psi^\dagger\tau\rho_1\psi \quad \text{for pseudoscalar mesons} \tag{1.3b}$$

ψ is the wave function of nucleons, U is the scalar or pseudoscalar wave function of mesons, A_0 and $\boldsymbol{A}$ are the scalar and vector potentials of photons $(A_0=0)$, g_1, g_2, f_1 and f_2 are the coupling constants,* μ is the mass of a meson, the units are so chosen that $\hbar=c=1$, and the dagger means Hermitian conjugate matrix or operator. Other notations are all usual. When we consider neutral mesons the similar expression is to be added appropriately omitting e-terms.

By the Legendre transformation of this Lagrangian function we have the Hamiltonian operator of the system as follows:

$$H=H^n+H^m+H^e+H^{ne}+H^{me}+H^{nm}+H^{nme}$$

$$+\text{second degree terms in the coupling constants} \tag{1.4}$$

where the upper suffices n, m, and e mean nucleons, mesons, and photons, respectively. The interactions are given by

$$H^{me}=e\int\psi^\dagger\tau^\dagger\tau(A_0-\rho_1\sigma\boldsymbol{A})\psi \tag{1.5a}$$

$$H^{me}=\frac{ie}{4\pi}\int\{\boldsymbol{A}(U^\dagger grad\,U-U\,grad\,U^\dagger)+A_0(U^\dagger F-UF^\dagger)\} \tag{1.5b}$$

$$H^{nm}=\int(QU^\dagger+Q^\dagger U)+\frac{1}{\mu}\int(\boldsymbol{P}\,grad\,U^\dagger+\boldsymbol{P}^\dagger\,grad\,U+P_0F^\dagger+P_5^\dagger F) \tag{1.5c}$$

$$H^{nme}=\frac{ie}{\mu}\int\boldsymbol{A}(\boldsymbol{P}U^\dagger-\boldsymbol{P}^\dagger U) \tag{1.5d}$$

where $F^\dagger/4\pi$ is the canonical conjugate of U, and the integral signs mean the space integrals. If we define H_0 and S (a skew Hermitian operator) by

$$H_0=H^n+H^m \tag{1.6a}$$

$$S=\frac{i}{\mu}\int(P_0^\dagger U+P_0U^\dagger) \tag{1.6b}$$

we have, by virtue of the commutation relations which are satisfied by ψ, U, and F, the following relations:

$$H^{nm}=H_1+[H_0S] \tag{1.7a}$$

$$H^{nme}=[H^{ne}S]+[H^{me}S] \tag{1.7b}$$

where the square bracket denotes the commutator, H_1 is given by

$$H_1=\int(R^\dagger U+RU^\dagger) \tag{1.8a}$$

$$\left.\begin{array}{ll} R=Q & \text{for scalar mesons} \\ R=(f_3/f_1)Q,\ f_3=f_1+(2M/\mu)f_2, & \text{for pseudoscalar mesons} \end{array}\right\} \tag{1.8b}$$

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* $\sqrt{2}\ f$ in A is equal to g or f in the present paper.

and M is the mass of a nucleon.

We now transform the Hamiltonian into the form $(\exp S)H\exp(-S)$. This transform is, by dint of (1.7), represented as follows:

$$H=H^{n}+H^{m}+H^{e}+H^{ne}+H^{me}+H_1$$
$$+\text{higher degree terms in the coupling constants} \qquad (1.9)$$

Thus we have the first degree interaction, between nucleons and mesons, of the scalar or pseudoscalar type only.*9),10),11) Further if we once again transform the Hamiltonian (1.9) into the form $(\exp S_1)H\exp(-S_1)$ where S_1 (a skew Hermitian operator) is defined by

$$H_0S_1]=H_1 \qquad (1.10)$$

the transform then takes the following form:

$$H=H^{n}+H^{m}+H^{e}+H^{ne}+H^{me}+[S_1H^{ne}]+[S_1H^{me}]$$
$$+\text{higher degree terms in the coupling constants} \qquad (1.11)$$

§ 2. Differential Cross Section

We consider the processes (i) $\gamma+P\rightarrow N+\pi^+$, (ii) $\gamma+N\rightarrow P+\pi^-$, and (iii) $\gamma+P\rightarrow P+\pi^0$ where γ, P, N, and π denote a photon, a proton, a neutron, and a meson respectively, and the nucleon is assumed to be initially at rest. The upper suffices +, −, and o mean positive, negative, and neutral mesons respectively. The general expression of the differential cross section for the process was obtained in A.3) It is given by

$$d\sigma=2\pi\sum\overline{|\Omega|^2}\,EWdE/E_0 \qquad (2.1)$$

where E_0, E, and W are the energy of the incident photon, the emitted meson, and the recoil nucleon respectively, Ω is $V/2\pi$ times the matrix element of the interaction energy, $\sum$ means the sum over spin indices of the recoil nucleon, the upper bar denotes an average over polarizing directions of the incident photon and initial spin states of the nucleon, and V is the volume of the normalization space.

Since we are interested only in $(eg)^2$ or $(ef)^2$ terms the process is due to $[S_1H^{ne}]$ and $[S_1H^{me}]$ in (1.11), and $2\pi\Omega/V$ is equal to the matrix element of $[S_1(H^{ne}+H^{me})]$ between the initial and final states. If we calculate on the basis of the original expression of the Hamiltonian given by (1.4) as was carried out in A, the result is exactly the same.

Positive mesons. For the process $\gamma+P\rightarrow N+\pi^+$, after the spin summation over the intermediate states we have Ω of (2.1) as follows:

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* The equivalence of the pseudovector coupling to the pseudoscalar one was already known in A so far as our process was concerned.

$$\Omega_s^+ = \frac{eg_1}{\sqrt{E_0 E}}\left\{\frac{(E_0-\rho_1\sigma l)\rho_1\sigma e}{2ME_0} + \frac{2ke}{2M(E_0-E)+\mu^2}\right\} \tag{2.2a}$$

$$\Omega_{ps}^+ = \frac{ef_3}{\sqrt{E_0 E}}\left\{\frac{(E_0-\rho_1\sigma l)i\sigma e}{2ME_0} + \frac{2ke\rho_2}{2M(E_0-E)+\mu^2}\right\} \tag{2.2b}$$

where the suffices *s* and *ps* distinguish the cases of scalar and pseudoscalar mesons, the expressions on the right are the abbreviation of their matrix elements between the initial and final spin states of the nucleon, $\boldsymbol{e}$ and $\boldsymbol{l}$ are the polarization (unit) vector and momentum of the incident photon respectively, and $\boldsymbol{k}$ is the momentum of the emitted meson. The first term comes from $[S_1 H^{ne}]$ and the second comes from $[S_1 H^{me}]$. It can be shown that (2.2b) is just the same as (2.10) of *A*.

The differential cross section can be obtained according to the method which was already explained in *A*. The exact expressions of it were given by (2.17) and (2.23) in *A*. It can be written in the following form:

$$d\sigma_s^+ = a_s\phi_s d\varepsilon/\varepsilon_0^2 \qquad d\sigma_{ps}^+ = a_{ps}\phi_{ps} d\varepsilon/\varepsilon_0^2 \tag{2.3}$$

$$a_s = \frac{\pi}{2}\left(\frac{eg_1}{M}\right)^2 \qquad a_{ps} = \frac{\pi}{2}\left(\frac{ef_3}{M}\right)^2 \qquad f_3 = f_1 + \frac{2M}{\mu}f_2 \tag{2.4}$$

$$\phi_s = \frac{\varepsilon - b^2/2}{\varepsilon_0} + (4-b^2)\frac{(\varepsilon^2-b^2)\sin^2\theta}{2(\varepsilon_0+b^2/2-\varepsilon)^2} \tag{2.5a}$$

$$\phi_{ps} = \frac{\varepsilon - b^2/2}{\varepsilon_0} - b^2\frac{(\varepsilon^2-b^2)\sin^2\theta}{2(\varepsilon_0+b^2/2-\varepsilon)^2} \tag{2.5b}$$

$$\varepsilon = E/M \qquad \varepsilon_0 = E_0/M \qquad b = \mu/M \tag{2.6}$$

where θ is the angle between directions of the incident photon and the emitted meson, and the relation between θ and E was given by (2.3) of *A*.

Negative Mesons. For the process $\gamma + N \rightarrow P + \pi^-$, Ω of (2.1) is given by

$$\Omega_s^- = -\frac{eg_1}{\sqrt{E_0 E}}\left\{\frac{\rho_1\sigma e(\rho_1\sigma k + E - 2M)}{2ME-\mu^2} + \frac{2ke}{2M(E_0-E)+\mu^2}\right\} \tag{2.7a}$$

$$\Omega_{ps}^- = -\frac{ef_3}{\sqrt{E_0 E}}\left\{\frac{i\sigma e(\rho_1\sigma k + E)}{2ME-\mu^2} + \frac{2ke\rho_2}{2M(E_0-E)+\mu^2}\right\} \tag{2.7b}$$

The first term comes from $[S_1 H^{ne}]$ and the second comes from $[S_1 H^{me}]$. The former is different from whereas the latter is the same as those in the positive meson case except for its sign.

We can change the numerators of the first terms into the form in which those of the positive meson case are involved.* Making use of the commutation relation

$$(\sigma e)(\sigma k) + (\sigma k)(\sigma e) = 2ke, \tag{2.8}$$

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* The author thanks Messrs. Koba, Kotani and Nakai for their valuable discussion in this point.

the conservation relations $\boldsymbol{l}=\boldsymbol{p}+\boldsymbol{k}$ and $M+E_0=W+E$, and the Dirac equations $\rho_3=1$ and $\rho_1\boldsymbol{\sigma p}+M\rho_3=W\rho$ respectively for the wave functions of the initial and final states of the nucleon, we get

$$\rho_1\boldsymbol{\sigma e}(\rho_1\boldsymbol{\sigma k}+E-2M)=(E_0-\rho_1\boldsymbol{\sigma l})\rho_1\boldsymbol{\sigma e}+2\boldsymbol{ke} \tag{2.9a}$$

$$i\boldsymbol{\sigma e}(\rho_1\boldsymbol{\sigma k}+E)=(E_0-\rho_1\boldsymbol{\sigma l})i\boldsymbol{\sigma e}+2\boldsymbol{ke}\rho_2 \tag{2.9b}$$

where $\boldsymbol{p}$ is the momentum of the nucleon in the final state. From (2.2), (2.7) and (2.9) we can find the following relation:

$$-\left(\frac{\Omega^-}{\Omega^+}\right)_s=-\left(\frac{\Omega^-}{\Omega^+}\right)_{ps}=\frac{2ME_0}{2ME-\mu^2} \tag{2.10}$$

The differential cross section can therefore be obtained by multiplying a square of this ratio into $d\sigma^+$:

$$\left(\frac{d\sigma^-}{d\sigma^+}\right)_s=\left(\frac{d\sigma^-}{d\sigma^+}\right)_{ps}=\left(\frac{\varepsilon_0}{\varepsilon-b^2/2}\right)^2=\left(\frac{E_0}{E-\mu^2/2M}\right)^2 \tag{2.11}$$

Neutral Mesons. For the process $\gamma+P\to P+\pi^0$, $[S_1H^{ne}]$ only is responsible, and Ω is given by a sum of the first terms of (2.2) and (2.7):

$$\Omega_s^0=\frac{eg_1^{\,0}}{\sqrt{E_0E}}\left\{\frac{(E_0-\rho_1\boldsymbol{\sigma l})\rho_1\boldsymbol{\sigma e}}{2ME_0}-\frac{\rho_1\boldsymbol{\sigma e}(\rho_1\boldsymbol{\sigma k}+E-2M)}{2ME-\mu_0^{\,2}}\right\} \tag{2.12a}$$

$$\Omega_{ps}^0=\frac{ef_3^{\,0}}{\sqrt{E_0E}}\left\{\frac{(E_0-\rho_1\boldsymbol{\sigma l})i\boldsymbol{\sigma e}}{2ME_0}-\frac{i\boldsymbol{\sigma e}(\rho_1\boldsymbol{\sigma k}+E)}{ME-\mu_0^{\,2}}\right\} \tag{2.12b}$$

where $f_3^{\,0}=f_1^{\,0}+(2M/\mu)f_2^{\,0}$, and μ_0, $g_1^{\,0}$, $f_1^{\,0}$, and $f_2^{\,0}$ are the mass of neutral meson, the constants of the scalar, pseudoscalar, and pseudovector couplings of neutral mesons with nucleons respectively.

If we change their form by making use of the relations given by (2.9) we can find the following relation:

$$-\left(\frac{\Omega^0}{\Omega^+}\right)_s=-\left(\frac{\Omega^0}{\Omega^+}\right)_{ps}=\frac{2M(E_0-E)+\mu^2}{2ME-\mu^2} \tag{2.13}$$

Therefore the differential cross section can be obtained at once by multiplying a square of this ratio into $d\sigma^+$:

$$\left(\frac{d\sigma^0}{d\sigma^+}\right)_s=\left(\frac{d\sigma^0}{d\sigma^+}\right)_{ps}=\left(\frac{\varepsilon_0+b^2/2-\varepsilon}{\varepsilon-b^2/2}\right)^2=\left(\frac{E_0+\mu^2/2M-E}{E-\mu^2/2M}\right)^2 \tag{2.14}$$

where $\mu_0=\mu$, $g_1^{\,0}=g_1$, $f_1^{\,0}=f_1$, and $f_2^{\,0}=f_2$ are assumed.

Negative or Neutral to Positive Ratio. For $\theta=90°$, by virtue of the relation between θ and E which was given by (2.3) of *A*, the ratios given by (2.11) and (2.14) become

$$\left(\frac{d\sigma^-}{d\sigma^+}\right)_{\theta=90°}=\left(\frac{1+E_0/M}{1-\mu^2/2M^2}\right)^2 \tag{2.15a}$$

$$\left(\frac{d\sigma^0}{d\sigma^+}\right)_{\theta=90°}=\left(\frac{E_0/M+\mu^2/2M^2}{1-\mu^2/2M^2}\right)^2 \tag{2.15b}$$

For $\mu/M=286/1840$ the numerical values of these ratios are given by the following table :

ε_0	0.20	0.24	0.28	0.32	0.36	0.5	1
E_0 in Mev	188	226	263	301	339	470	940
$(d\sigma^-/d\sigma^+)_{\theta=90°}$	1.48	1.58	1.68	1.79	1.90	2.30	4.10
$(d\sigma^0/d\sigma^+)_{\theta=90°}$	0.046	0.065	0.087	0.113	0.142	0.268	1.05

The relation of the angular distribution and the negative to positive ratio with experiment was already discussed by Brueckner and Goldberger,[7] and Koba, Kotani and Nakai.[8]

§ 3. Total Cross Section

The maximum and minimum values of E was given by (2.19) of A. The total cross sections can be obtained by integrating the differential cross section from the minimum E to the maximum E where $\sin\theta$ is replaced by a function of E according to the relation given by (2.3) of A. The differential cross sections consist of two terms as is seen from (2.5), (2.11), and (2.14). Corresponding to them we have the total cross sections which consist of two terms :

$$\sigma_s^+=a_s\{\xi^+ + (4-\delta^2)\eta^+\} \tag{3.1a}$$

$$\sigma_s^-=a_s\{\xi^- + (4-\delta^2)\eta^-\} \tag{3.1b}$$

$$\sigma_s^0=a_s^0\{\xi^0 + (4-\delta^2)\eta^0\} \tag{3.1c}$$

$$\sigma_{ps}^+=a_{ps}(\xi^+ -\delta^2\eta^+) \tag{3.2a}$$

$$\sigma_{ps}^-=a_{ps}(\xi^- -\delta^2\eta^-) \tag{3.2b}$$

$$\sigma_{ps}^0=a_{ps}^0(\xi^0 -\delta^2\eta^0) \tag{3.2c}$$

The terms containing ξ's are the integrals of the first terms, and those containing η's are the integrals of the second terms. ξ's and η's are obtained by elementary integrations of the first and second terms as follows :

$$\xi^+=\frac{2(1+\varepsilon_0-\delta^2/2)}{\varepsilon_0(1+2\varepsilon_0)^2}\zeta \tag{3.3a}$$

$$\xi^-=(1/\varepsilon_0)\log\Psi \tag{3.3b}$$

$$\xi^0=\frac{1}{\varepsilon_0}\log\Psi-\frac{2(1+3\varepsilon_0+\delta^2/2)}{\varepsilon_0(1+2\varepsilon_0)^2}\zeta \tag{3.3c}$$

$$\eta^+=\{(\varepsilon_0+\delta^2/2)\log\Phi^+-2\zeta\}/\varepsilon_0^3 \tag{3.4a}$$

$$\eta^- = \{(\varepsilon_0 - b^2/2)\log \Phi^- - 2\zeta\}/\varepsilon_0^3 \tag{3.4b}$$

$$\eta^0 = \{(1+\varepsilon_0 - b^2/2)\log \Psi - 2\zeta\}/\varepsilon_0^3 \tag{3.4c}$$

where μ in ξ^0 and η^0 has to be understood as μ_0, and

$$\zeta = \sqrt{(\varepsilon_0 - b^2/2)^2 - b^2} \tag{3.5}$$

$$\Phi^+ = \frac{\varepsilon_0 + b^2/2 + \zeta}{\varepsilon_0 + b^2/2 - \zeta} \tag{3.6a}$$

$$\Phi^- = \frac{\varepsilon_0 - b^2/2 + \zeta}{\varepsilon_0 - b^2/2 - \zeta} = \Psi\,\Phi^+ \tag{3.6b}$$

$$\Psi = \frac{1+\varepsilon_0 - b^2/2 + \zeta}{1+\varepsilon_0 - b^2/2 - \zeta} \tag{3.6c}$$

The threshold energy is given by $(\varepsilon_0)_t = b + b^2/2$. σ's all vanish at this energy. If η's are expanded in power series in ζ their first terms are proportional to ζ^3. Therefore in the immediate vicinity of threshold σ's are equal to ξ's. Their first terms in power series in ζ are given by

$$\left(\frac{\sigma^+}{a}\right)_s = \left(\frac{\sigma^+}{a}\right)_{ps} = \frac{2\sqrt{(\varepsilon_0 - b^2/2)^2 - b^2}}{\varepsilon_0(1+b)^3} \tag{3.7}$$

$$\left(\frac{\sigma^-}{a}\right)_s = \left(\frac{\sigma^-}{a}\right)_{ps} = \frac{2\sqrt{(\varepsilon_0 - b^2/2)^2 - b^2}}{\varepsilon_0(1+b)} \tag{3.8}$$

$$\left(\frac{\sigma^0}{a}\right)_s = \left(\frac{\sigma^0}{a}\right)_{ps} = b_0^2\,\frac{2\sqrt{(\varepsilon_0 - b_0^2/2)^2 - b_0^2}}{\varepsilon_0(1+b_0)^3} \tag{3.9}$$

All of them are proportional to 1/2 power of the excess energy over the threshold. The ratio of the production of negative to positive mesons near threshold is given by

$$(\sigma^-/\sigma^+)_s = (\sigma^-/\sigma^+)_{ps} = (1+\mu/M)^2 \tag{3.10}$$

If we adopt $\mu/M = 286/1840$ we have $\sigma^-/\sigma^+ = 1.33$ for the threshold, whereas we have $\sigma^- = \sigma^+$ in the limit $M \to \infty$. Therefore we can not consider the nucleon to be infinitely heavy even near threshold, as was considered by Foldy.[6)]

The ratio for the production of neutral and charged mesons near threshold is given by

$$(\sigma^0/\sigma^+)_s = (\sigma^0/\sigma^+)_{ps} = (\mu/M)^2 = 0.024 \tag{3.11}$$

where $a^0 = a$, $\mu_0 = \mu$ and $\mu/M = 286/1840$ are assumed. This is very small. The reason is that the production of neutral mesons is due to $[S_1 H^{ne}]$ only.

In order to examine the behavior of $\sigma(\varepsilon_0)$ in the high energy region, we consider their asymptotic expressions. If we omit all small terms for large ε_0 we have

$$\sigma_s^+ = a_s\left\{\frac{1}{2\varepsilon_0} + \frac{4}{\varepsilon_0^2}\left(\log\frac{2\varepsilon_0}{b^2} - 2\right)\right\} \tag{3.12a}$$

$$\sigma_s^- = a_s\left\{\frac{1}{\varepsilon_0}\log 2\varepsilon_0 + \frac{8}{\varepsilon_0^2}\left(\log\frac{2\varepsilon_0}{b} - 1\right)\right\} \tag{3.12b}$$

$$\sigma_s^0 = a_s\left\{\frac{1}{\varepsilon_0}\left(\log 2\varepsilon_0 - \frac{3}{2}\right) + \frac{4}{\varepsilon_0^2}(\log 2\varepsilon_0 - 2)\right\} \tag{3.12c}$$

$$\sigma_{ps}^+ = a_{ps}/(2\varepsilon_0) \tag{3.13a}$$

$$\sigma_{ps}^- = a_{ps}(\log 2\varepsilon_0)/\varepsilon_0 \tag{3.13b}$$

$$\sigma_{ps}^0 = a_{ps}(\log 2\varepsilon_0 - 3/2)/\varepsilon_0 \tag{3.13c}$$

where the latter terms in (3.12) are due to η's and the terms containing ξ's are only retained in (3.13). The approximation of these asymptotic expressions is sufficient for ε_0 larger than about 50 (i.e. about 50 Bev of photon energy). (3.12a) and (3.13a) agree with (2.22) and (2.24) of *A*, but not with the results of Nordheim and Nordheim,[1] and Yukawa and others.[2]

The dependence of the total cross section on the energy of incident photons

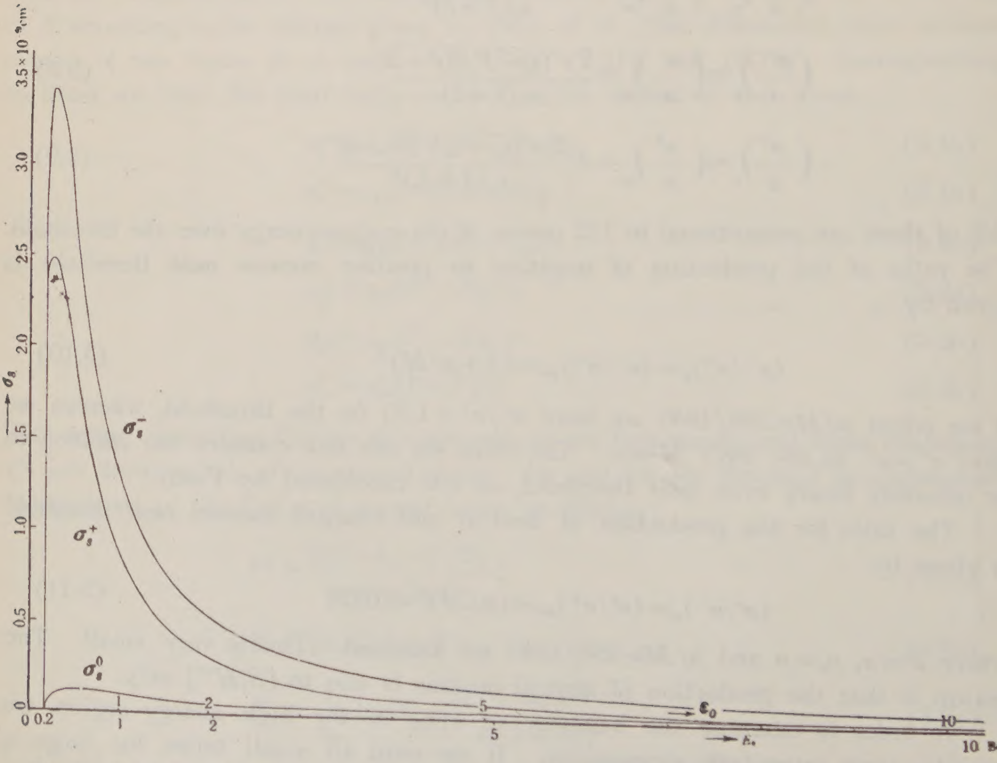

Fig. 1. Total Cross Section for the Production of Scalar Mesons by Photons. (unit of $\sigma = 10^{-29}\,\text{cm}^2$)

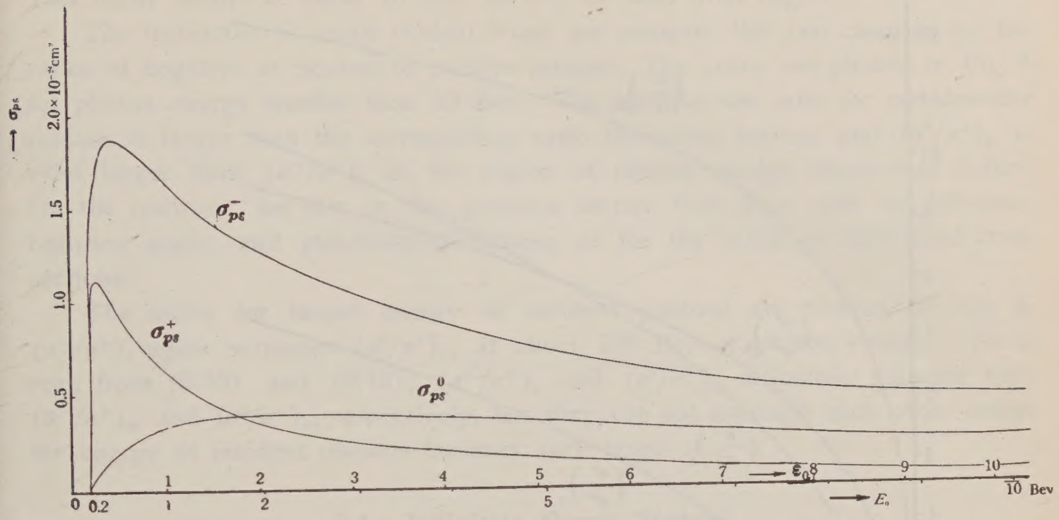

Fig. 2. Total Cross Section for the Production of Pseudoscalar Mesons by Photons. (unit of $\sigma=10^{-28}$ cm²)

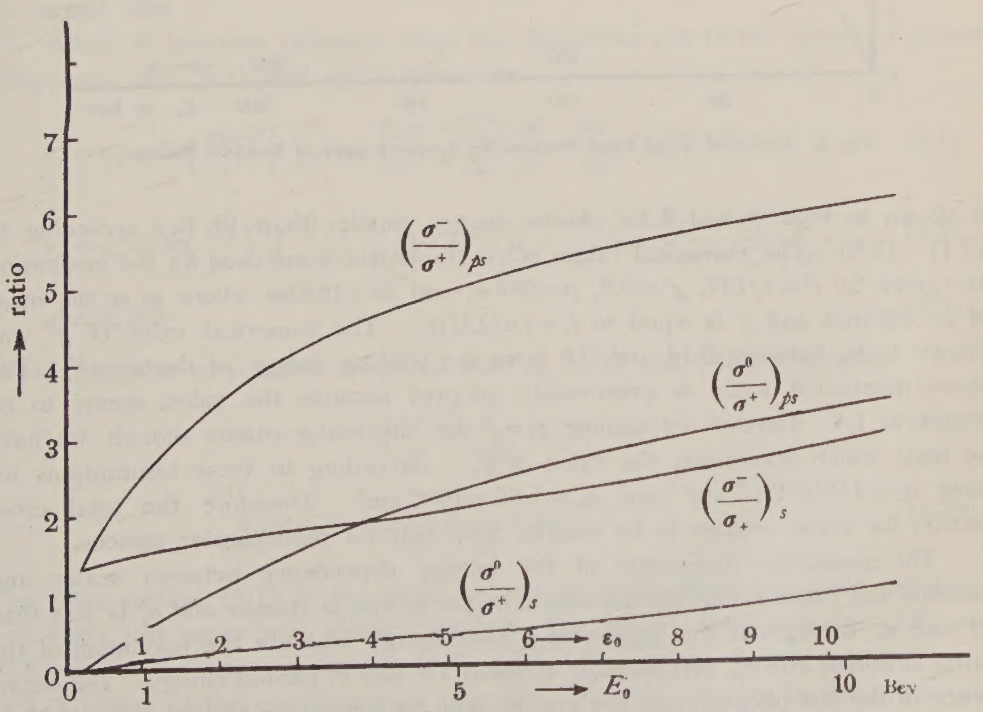

Fig. 3. Ratios of Total Cross Sections for Photon Energy smaller than 10 Bev.

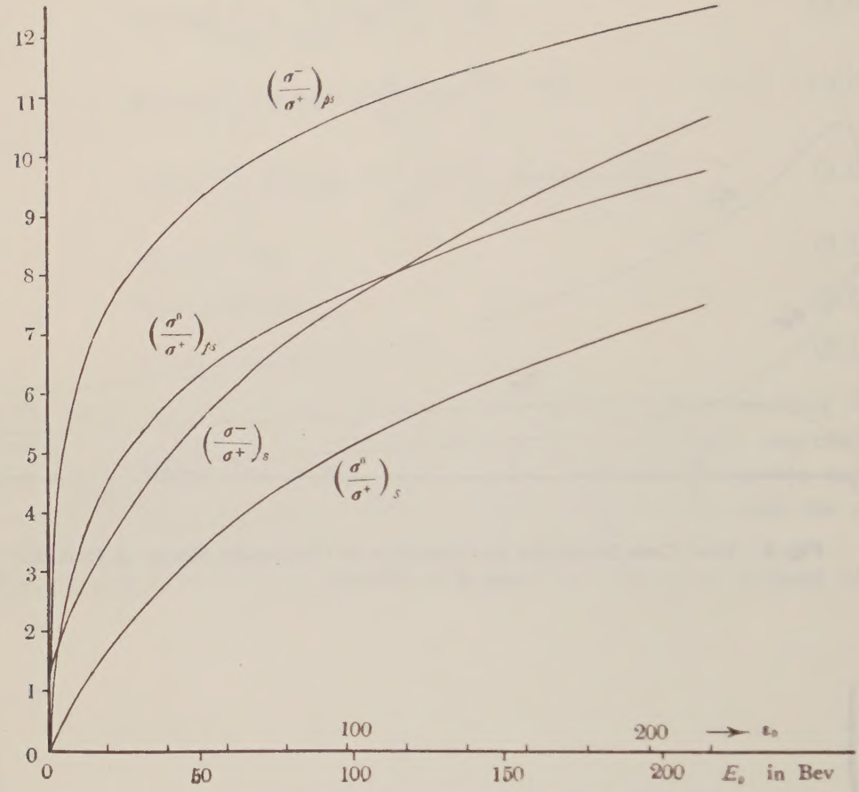

Fig. 4. Ratios of Total Cross Sections for Larger Energy of Incident Photons.

is shown in Figs. 1 and 2 for photon energy smaller than 10 Bev according to (3.1)—(3.6). The numerical values of constants which are used for the evaluation are given by $e^2=1/137$, $g^2=0.2$, $\mu=286m$, and $M=1840m$ where m is the mass of an electron and g is equal to $f_2+(\mu/2M)f_1$. The numerical value of g^2 was known to be between 0.14 and 0.5 from the binding energy of deuteron.[12] The above mentioned value is presumably adopted because the value seems to be nearer to 1.4. Further we assume $g_1=g$ for the scalar meson though we have no basis which determines the value of g_1. According to these assumptions we have $a_s=1.00\times10^{-30}$ cm^2 and $a_{ps}=1.66\times10^{-28}$ cm^2. Therefore the total cross section for scalar mesons is far smaller than that for pseudoscalar mesons.

The qualitative distinction of the energy dependence between scalar and pseudoscalar cases is that the maximum of the former is steeper and σ_s^0 is less than σ_s^+ and σ_s^- throughout this region of photon energy whereas the maximum of the latter is obtuse and σ_{ps}^0 surpasses σ_{ps}^+ at about 1.8 Bev of photon energy. The occurrence of the last phenomenon can also be seen by comparing (3.13a) with (3.13c). Further it can be seen from (3.12) that σ_s^0 surpasses σ_s^+ too at higher energy.

This really occurs at about 10 Bev, as will be seen from Fig. 4.

The distinction is more evident when we compare the two cases as for the ratios of negative or neutral to positive mesons. The ratios are plotted in Fig. 3 for photon energy smaller than 10 Bev. We see that the ratio for pseudoscalar mesons is larger than the correspoding ratio for scalar mesons and $(\sigma^0/\sigma^+)_{ps}$ is even larger than $(\sigma^-/\sigma^+)_s$ in the region of photon energy larger than 4 Bev. On the contrary, we saw in the previous section that there was no difference between scalar and pseudoscalar mesons as for the ratios of differential cross sections.

The ratios for larger energy of incident photons are plotted in Fig. 4. $(\sigma^-/\sigma^+)_s$ again surpasses $(\sigma^0/\sigma^+)_{ps}$ at about 110 Bev of photon energy. As is seen from (3.12) and (3.13), $(\sigma^-/\sigma^+)_s$ and $(\sigma^0/\sigma^+)_s$ ultimately coincide with $(\sigma^-/\sigma^+)_{ps}$ and $(\sigma^0/\sigma^+)_{ps}$ respeetively, but they can not approach each other unless the energy of incident photons becomes very large.

§ 4. Infinitely Heavy Nucleon

In order to know the formal nature of the result and to compare it with other author's we consider a limit of infinitely heavy nucleon in this last section. Of course, M is finite in fact, and we can not apply the result of this section to the actual case.

When M becomes infinitely large the distinction due to the charge of mesons disappears, and (3.1) and (3.2) reduce to

$$\sigma_s^+=\sigma_s^-=2\pi\left(\frac{eg_1}{\mu}\right)^2\left\{\frac{\mu^2}{E_0^2}\log\frac{E_0+\sqrt{E_0^2-\mu^2}}{E_0-\sqrt{E_0^2-\mu^2}}-\frac{2\mu^2}{E_0^3}\sqrt{E_0^2-\mu^2}\right\} \tag{4.1}$$

$$\sigma_{ps}^+=\sigma_{ps}^-=2\pi\left(\frac{ef_2}{\mu}\right)^2\left\{\left(\frac{2}{E_0}+\frac{2\mu^2}{E_0^3}\right)\sqrt{E_0^2-\mu^2}-\frac{\mu^2}{E_0^2}\log\frac{E_0+\sqrt{E_0^2-\mu^2}}{E_0-\sqrt{E_0^2-\mu^2}}\right\} \tag{4.2}$$

$$\sigma_s^0=\sigma_{ps}^0=0 \tag{4.3}$$

When $E_0-\mu$ is very small compared with μ, (4.1) and (4.2) reduce to

$$\sigma_s^+=\sigma_s^-=4\pi\left(\frac{eg_1}{\mu}\right)^2\frac{\mu^2}{3E_0^5}\sqrt{(E_0^2-\mu^2)^3} \tag{4.4}$$

$$\sigma_{ps}^+=\sigma_{ps}^-=4\pi\left(\frac{cf_2}{\mu}\right)^2\frac{1}{E_0}\sqrt{E_0^2-\mu^2} \tag{4.5}$$

The cross section (4.1) is the same as Nordheim-Nordheim's result[1] and (4.1) and (4.2) just agree with Foldy's result.[6] The energy dependence of (3.7) and (3.8) is not the same as that of (4.4) and (4.5). In the former case both vary as 1/2 power of the excess energy over the threshold whereas in the latter case σ_s varies as 3/2 power and σ_{ps} varies as 1/2 power. This disagreement is due to the fact that a_s varies as $1/M$ when $M\to\infty$ whereas a_{ps} approaches to a non-vanishing limit as is seen from (2.4).

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Meson Fields and Nuclear Forces

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(Received May 25, 1950)

He thought he saw a Banker's Clerk
Descending from the bus :
He looked again, and found it was
A Hippopotamus :
" If this should stay to dine", he said,
" There won't be much for us ! "

These whimsical lines of Lewis Carroll* forcibly come to my mind as I begin writing this essay and glance back at the vicissitudes which our ideas about mesons have been undergoing in rapid succession as a result of the experimenters' " looking again " at the tracks in their cloud chambers and emulsion plates. This bewildering development may at times drive theorists to the feeling of despondency expressed in the last line of the poem, although the complementary statement " this is too much for us " might seem even more appropriate.

Nevertheless, it is remarkable - and it must give Professor Yukawa legitimate satisfaction—to see how his original conception of the meson field as the mechanism of nuclear interactions has emerged unscathed from the turmoil. *Post festum* it is easy, of course, to discover good reasons for this, and perhaps a bit pedantic to point them out. The most fundamental is doubtless the simple and direct connexion of the meson field conception of nuclear forces with the relationship of complementarity between field and particle, finding its quantitative expression in the formula which defines the range of the interaction transmitted by the field in terms of the mass of the associated particle. Yukawa's assumption that the transmission of the force is effected by a single quantum, and that consequently the mesons have integral spin, though not the only one that suggested itself at the time, was certainly the most straightforward, and it has now been fully vindicated by the Berkeley results. The alternative possibility was a theory in which the interactions are transmitted by pairs of quanta : in view of its unattractive features, its elimination will probably be felt as a relief by theoretical physicists.

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* LEWIS CARROLL, *Sylvie and Bruno*, chapter VII (1889).

This does not mean, however, that all is well in meson theory. Its divergences have not yielded to the methods of renormalization and regularization which have lately proved so powerful in electrodynamics. It may well be that the root of the enhanced difficulty lies in the essentially non-linear character of meson theory, exemplified by the preponderance of multiple processes of meson production at very high energies. It is even conceivable that the features of non-linearity remain predominant in nuclear interactions, and in particular (as suggested by Teller) essentially determine the saturation properties of these interactions.

The present trend of empirical evidence about nuclear forces, although perhaps not so definite as one might wish, does not seem to me to point to such a radical conclusion. The consequences of meson field theory in the linear approximation have not yet been sufficiently explored beyond the static case, and recent work seems to indicate that study on such lines is certainly worth while. I think we ought not to allow ourselves to be frightened by the pertinacious divergences encountered at every step: any rough and ready cut-off procedure will at any rate give us the general orientation that we want in the first instance.

As regards the question how far results obtained by cutting-off divergent integrals can be trusted, some calculations on the deuteron problem may be relevant. In this case, one may ask whether the cut-off of the static potential can be regarded as a roughly correct way of accouting for the effect of non-static corrections at small distances. Now, this rather optimistic surmise has been strikingly confirmed by a more accurate discussion of the non-static effects on the basis of Van Hove's representation of the interaction in momentum space, which includes the second order terms in the nucleon velocities. It appears that in this particular instance even a severe cut-off of the static potential leads to the same qualitative picture as the more exact treatment.

These calculations disclose a somewhat unexpected property of the type of interaction defined by meson fields. It is found that, in marked contrast with the case of a "well" shape of the nuclear potential discussed by Rarita and Schwinger, the non-central part of the interaction contributes relatively little to the binding of the ground state. This property, the consequences of which have not yet been examined in detail, may be expected to contribute to the elucidation of some obscure points, such as the effect of non-central couplings on the saturation properties of nuclear binding. This is only one of several features of the interaction between nucleons which have been recognized to depend critically on the shape of the nuclear potential.

In fact, the data concerning the low energy states of the two-nucleon system, derived from the study of the scattering of neutrons and protons by protons and of the diffraction of very slow neutrons, yield interesting information in this respect. When these data are analyzed by the ingenious method due to Schwinger, it is found that owing to their comparatively low accuracy, they can be fitted in principle to any type of potential. But if the well shape is used, quite different ranges (and depths) must be assumed for the effective potentials corresponding to the ^3S and ^1S states; if, however, a potential of the meson type is assumed, it is possible to account for all data with a single range constant for the ^3S and ^1S states. This result certainly provides a strong argument in favour of a shape of potential characterized by a long "tail", i.e. a relatively slow decrease at large distances. Quantitatively, the range value derived from the data on the assumption of a meson potential is in good agreement with the mass value of the π-mesons.

The conclusions regarding the form of nuclear interactions, deduced from the properties of two-nucleon systems, may have an important bearing on a problem of great interest which is still *sub judice*, namely, the form of the interaction between two α-particles. It is known that this property is also very sensitive to the "tail" of the two-nucleon potential. According to the more or less abrupt decrease of the latter at large distances, the predominant interaction between the α-particles is of the complicated charge exchange type or of the simpler Van der Waals form. A detailed investigation of the problem in the light of the preceding considerations would well repay the trouble.

A further important result of the above-mentioned analysis of the data on two-nucleon systems is the confirmation of the charge-independence property of the nuclear interaction, at any rate for the ^1S states. From the point of view of meson theory, this implies the necessity of postulating the existence of neutral mesons of the same kind as the charged ones. A consequence of the coupling of such neutral mesons to nucleons, early pointed out by Sakata and Tanikawa, is their spontaneous decay within an extremely short time into γ-rays. The brilliant experimental confirmation at Berkeley of these theoretical predictions is a triumph for the meson field idea hardly less resounding than was at the time the first discovery of the (wrong) mesons in cosmic radiation. It has indeed far-reaching implications, not only with respect to the occurrence of negative protons, but also regarding the spin of the π-mesons. In fact, we are now in a position to state definitely that the neutral mesons hitherto observed, and consequently at least part of the charged mesons, have the spin zero.

Whether there are also π-mesons of spin one is a vexed question, about which we are still very much in the dark. It is sometimes stated that their existence is unlikely, since the divergences of meson field theory are so much worse in the case of spin one than in that of spin zero. This reminds me of the man who had dropped his latchkey in a dark alley, but was looking for it in the illuminated area round a lamp post, since that was the only place where one could see anything. In a more solemn mood, this is the occasion to quote Fresnel's phrase: "La nature se joue de nos difficultés analytiques". Failing more or less direct evidence on the spin of the charged mesons observed either in cosmic radiation or at Berkeley, the only thing to do is to look for hints on this point in the analysis of nuclear forces from the point of view of meson theory.

The treatment of the deuteron problem on the assumption of a static potential of the form predicted by meson theory (with the inevitable cut-off at small distances) leads to a value of the strength ratio between the central and non-central parts which can be obtained only with the help of a mixture of fields of spins zero and one. This argument is admittedly rather weak in view of the cut-off, and the investigation of the problem using the more accurate velocity-dependent interaction has not yet progressed sufficiently far to give evidence on the question. As regards the data on neutron-proton scattering at high energies, it has only been ascertained up to now that no interaction resulting from a mixture of meson fields of spin zero can possibly account for them; but it is not yet known whether an admixture of spin one mesons would improve the situation.

There is, however, one feature of nuclear forces which has lately come into prominence and which might be relevant to the question of the meson spins. It has been recognized that the kind of shell structure of the stable nuclei, revealed by the occurrence of configurations with "magic numbers" of protons or neutrons corresponding to enhanced stability, finds its simplest interpretation on the assumption that strong spin-orbit couplings between the constituent nucleons are at play. It has also been pointed out that a coupling of this type seems necessary to account for the scattering of very fast protons by protons. Now, there is only one type of meson field, the vector field, which can give rise to a sufficiently large spin-orbit interaction. If, moreover, further investigation confirms the present indications as to the relative smallness of the contributions from the axial dipole couplings, it would be conceivable that an admixture of vector meson field would just make itself conspicuous by a large spin-orbit interaction.

I am aware that the preceding review of the problems confronting the meson theory of nuclear forces must give the superficial impression that there is much more of perhaps reckless hope and surmise in it than of actual achievement. Behind this haze of speculation about future research, one should not, however, lose sight of the fundamental fact that, in a period of most exciting and unexpected discoveries, Yukawa's original conception of the meson has afforded the most reliable guidance for the unravelling of the new evidence and the firmest foundation for the building-up of its interpretation. Besides, meson theory is only fifteen years old, and is this not the age of growth and of promise?

Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

The Yukawa Theory of Nuclear Forces in the Light of Present Quantum Theory of Wave Fields.

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(Received June 5, 1950)

The Yukawa theory of nuclear forces[1] has led to many successes and, owing to the present state of quantum theory, to some difficulties. Among the successes one remembers first the existence of the π-meson and the possibility of describing the spin dependency and the saturation of nuclear forces by means of simple vector fields or pseudo-scalar fields. Among the difficulties we mention the divergence of the interaction at small distances of the nucleons and the impossibility of getting the correct mass defect for heavy nuclei when one takes the constants of the Yukawa field from the mass defect of light nuclei.[2] Furthermore, the existence of closed neutron and proton shells in the nucleus[3] and the behaviour of the cross section for elastic collisions of nucleons at very high energies indicate, that the Yukawa potential is not correct at small distances of the nucleons.

These difficulties cannot be really solved yet; but the recent progress in quantum theory of wave fields[4] shows so clearly the way towards the solution of these problems, that it may be worth while to discuss this way, even if it is still too early to work it out in the mathematical details.

In the relativistic quantum theory of wave fields we have learned, that the divergent results arise from the singularities in the commutation function. Therefore the correct theory will have to start with a *regular* commutation function. This starting point leads to a number of problems, which have been dealt with recently in many papers.[5] We mention the most important results: The wave function that obeys a regular commutation rule, corresponds necessarily to several different types of elementary particles, not only to one type. This implies, that nucleons interact, as Bhabha[6] has suggested, not only by means of π-mesons but also by other types of particles, in such a way, that the singularity of the force at small distances will disappear. Furthermore, the two coordinated wave functions, that obey the regular commutation rule, cannot be hermitian conjugates in the ordinary sense.[5] This leads probably to a change in the hamiltonian formalism in the range of the "smallest length" l_0 ($l_0 \sim 10^{-13}$ cm), which corresponds to a lack of point-to-point causality, again tending to wash out singularities of the field.

Thereby already many of the difficulties may have disappeared. The potential inside a nucleus will now be rather smooth, certainly much smoother than one would expect from potentials of the type $\frac{1}{r}e^{-\varkappa r}$ or the corresponding tensor force potential. As a result there will be only small forces acting upon a nucleon inside a nucleus; it is only at the surface of the nucleus that the nucleons will be pulled back into the nucleus by strong forces. This explains quite naturally the existence of separated neutron and proton orbits and closed shells in the nucleus.

The order of the closed shells can be understood, according to Haxel, Jensen, Suess[5] and Göppert-Mayer,[6] from a strong spin-orbit coupling of every nucleon. Gaus[7] has shown that this strong spin-orbit coupling results under certain conditions immediately from the vector-meson theory of Yukawa. Therefore one may at this point conclude from the experiments, that at larger distances of two nucleons the symmetrical vector meson theory with the mass of the π-meson will probably give a fairly good approximation, while at smaller distances the higher masses will come into play and the deviations from the hamiltonian formalism will make the definition of a potential rather doubtful, as it was expected long ago from the concept of the "smallest length."[8]

The existence of neutral mesons, possibly of the scalar type, may produce forces without the property of saturation. This would explain naturally the rather large mass defects of heavy nuclei as compared to the mass defects of light nuclei. The observed saturation would then, as Teller[9] has suggested, probably be brought about by the non-linear interaction terms in the field equation, which prevent the Yukawa field to increase above a certain value.

Another difficulty for the vector-meson theory was the sign of the quadrupole-moment of the deuteron, which seemed to favour the pseudo-scalar rather than the vector theory. The quadrupole moment of the deuteron is determined by the tensor force, which depends strongly on the potential at small distances; the mass defect and the spin-orbit coupling depend more strongly on the outer part of the potential function. Therefore the higher masses and the deviation from Hamilton formalism may be decisive for the quadrupole moment of the deuteron, while the mass defect and the spin-orbit coupling are mainly produced by the vector field of the normal π-mesons.

Finally the cross-section for elastic collision of very fast nucleons will decrease more rapidly with increasing energy than one would expect from the Yukawa potential $\frac{1}{r}e^{-\varkappa r}$ and the corresponding tensor potential. One may express this mathematical result by stating, that the introduction of the "smallest length" l_0 in the primary commutation function leads also to a "largest force" of the order l_0^{-2} (or in ordinary units $\frac{\hbar c}{l_0^2}$), so that a momentum transfer of much more than $\frac{\hbar}{l_0}$ in an elastic collision will be a rather rare event. The collision of very energetic nucleons will instead as a rule lead to the creation of new particles, first of π-mesons and at still higher energies of other masses. The quantitative question at which energies the deviations from the simple Yukawa potential appear cannot yet be solved.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On the Connection between Spin and Statistics

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(Received June 12, 1950)

§ 1. Introduction and Summary.

In relativistically invariant quantized field theories the following conditions are fulfilled in the normal cases of half-integer spin connected with exclusion principle (Fermions) and of integer spin connected with symmetrical statistics (Bosons).

1. The vacuum is the state of lowest energy. So long as no interaction between particles is considered the energy difference between this state of lowest energy and the state where a finite number of particles is present is finite.
2. Physical quantities (observables) commute with each other in two space-time points with a space-like distance. (Indeed due to the impossibility of signal velocities greater than that of light, measurements at two such points cannot disturb each other.)
3. The metric in the Hilbert-space of the quantum mechanical states is positive definite. This guarantees the positive sign of the values of physical probabilities.

There seems to be agreement now about the necessity of all three postulates in physical theories. In earlier investigations[1] I have shown that in the abnormal cases of half-integer spin connected with symmetrical statistics and of integer spin connected with exclusion principle, which do not occur in nature, not all of the three mentioned postulates can be fulfilled in a relativistically invariant quantized field theory. In this older formulation of the theory for the abnormal cases the postulate (1) was violated for half integer spins and the postulate (2) for integer spins, while postulate (3) was always fulfilled. Meanwhile *Dirac*[2] had directed the attention to the possibilty of mathematical theories in which the postulate (3) is abandoned in favour of more general indefinite metrics in the space of quantum states. In this theory the sum of all probabilities which is conserved in the course of time contains in general also negative terms ("negative probabilities") and the square of "self-adjoint" operators (which replace the hermitian operators of the usual theory) can also have negative expectation values.

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[1] For the spin values 1/2 and 0: Ann. de l'Inst. Poincaré **6** (1936), 137.
For higher spin values: Phys. Rev. **58** (1940), 716; Compare also my report: Rev. Mod. Phys. **13** (1941), 203: in the following quoted as "I".

[2] P. A. M. Dirac, Proc. Roy. Soc. A, **180** (1942), 1; Compare also my report: Rev. Mod. Phys. **15** (1943), 175 which in the following is quoted as "II".

Recently *Feynman* in his "theory of positrons",[1] which does not use directly the concept of field quantization but more intuitive methods (which he proves to be equivalent with the former) made the important remark, that the abnormal case of Bosons with spin 1/2 and also of Fermions with spin 0 could be treated in a way similar to the normal case. Considering the effect of an external electromagnetic field (which can produce and annihilate pairs of positive and negative particles) on the initial vacuum, he derived however for the probability that the vacuum is left unchanged a value larger than unity in the abnormal case—in contrast to the expected value smaller than one for the normal case. In this paper it is shown for spin 1/2 (§ 2 and 3) and for spin 0 (§ 4) that Feynman's treatment of the abnormal cases is equivalent to a different mathematical formulation of the field quantization than which I earlier took into consideration: The new form of the theory for the abnormal cases *preserves the postulates (1) and (2)* (and also covariance of the theory with respect to charge-conjugation) but violates the postulate (3) introducing "negative probabilities" for states with an odd number of negative particles present. Feynman's result of a probability larger than one for the vacuum in such theories is then immediately understandable as the excess above unity of this probability has to compensate the negative probability of states, where one pair is generated. The non-physical character of these negative probabilities for the abnormal cases is also stressed by the circumstance that the vacuum expectation value of the square of the integral of a component of the current over a finite space-time region becomes negative in these cases (compare § 3, eq. (A 48) and (B 49)). In the e^2-approximation these vacuum expectation values are indeed very closely connected with the value of the deviation from unity for the probability of the original vacuum in an external electromagnetic field.

§ 2. Spin 1/2-particles (spinor fields) without interaction.

We start with a spinor field $\psi_\alpha(x)$ and its adjoint field $\bar{\psi}_\alpha(x)$ satisfying the Dirac equation (in natural units $\hbar=c=1$)

$$\left(\gamma^\nu \frac{\partial}{\partial x_\nu} + m\right)\psi=0, \qquad (1)$$

and

$$\bar{\psi}\left(\gamma^\nu \frac{\partial}{\partial x_\nu} - m\right)=0, \qquad (2)$$

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1) R. P. Feynman, Phys. Rev. **76** (1949), 749, see especially p. 756, compare also D. Rivier, Helv. Phys. Acta **22** (1949).

where as usual

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\delta_{\mu\nu}\,.$$

The general solution can be decomposed in proper vibrations corresponding to plane waves according to[1]

$$\psi_{\alpha}(x)=\frac{1}{\sqrt{V}}\sum_{k}\sum_{r=1,2}\{u_r(k)\,a^r_{\alpha}(k)\,e^{i(k,x)}+\hat{v}^r(k)\,b^r_{\alpha}(k)\,e^{-i(k,x)}\},\tag{3}$$

$$\bar{\psi}_{\alpha}(x)=\frac{1}{\sqrt{V}}\sum_{k}\sum_{r=1,2}\{\hat{u}_r(k)\,\bar{a}^r_{\alpha}(k)\,e^{-i(k,x)}+v_r(k)\,\bar{b}^r_{\alpha}(k)\,e^{+i(k,x)}\}.\tag{4}$$

The connection between the operators $\bar{\psi}_{\alpha}(x)$, $\hat{u}_r(k)$, $\hat{v}_r(k)$ with the hermitian conjugates[2] of $\psi_{\alpha}(x)$, $u_r(k)$, $v_r(k)$ respectively we are still leaving open but for the c-numbers $\bar{a}^r_{\alpha}(k)$, $\bar{b}^r_{\alpha}(k)$ we assume as usual for every k

$$\bar{a}^r=a^{*r}\gamma^4,\quad \bar{b}^r=b^{*r}\gamma^4.\tag{5}$$

In the four dimensional scalar product (k,x) the fourth component of k_{ν} is always defined to be $k_4=i\omega=+i\sqrt{\vec{k}^2+m^2}$. We are normalizing the c-numbers, a^r_{α}, b^r_{α} according to

$$\sum a^{*r}_{\rho}a^{s}_{\rho}=\delta_{rs},\quad \sum b^{*r}_{\rho}b^{s}_{\rho}=\delta_{rs},\tag{6}$$

for all values of k. Then, from the Dirac-equation there follow also the other relations

$$\sum_{r} a^r_{\alpha}\bar{a}^r_{\beta}=-\frac{1}{2\omega}(i\gamma^{\nu}k_{\nu}-m)_{\alpha\beta}\,,\tag{7}$$

$$\sum_{r} b^r_{\alpha}\bar{b}^r_{\beta}=+\frac{1}{2\omega}(-i\gamma^{\nu}k_{\nu}-m)_{\alpha\beta}\,.\tag{8}$$

From this one gets the connection with Schwinger's functions[3]

$$S_{\alpha\beta}(x)=\left(\gamma^{\nu}\frac{\partial}{\partial x_{\nu}}-m\right)_{\alpha\beta}\Delta(x)\;;\quad S^{(1)}_{\alpha\beta}(x)=\left(\gamma^{\nu}\frac{\partial}{\partial x_{\nu}}-m\right)_{\alpha\beta}\Delta^{(1)}(x),$$

by the equations

$$\frac{1}{V}\sum_{k}\sum_{r} a^r_{\alpha}\bar{a}^r_{\beta}\,e^{i(k,x-x')}=-iS^{+}_{\alpha\beta}(x-x')=-\frac{i}{2}(S-iS^{(1)})(x-x'),\tag{9}$$

$$\frac{1}{V}\sum_{k}\sum_{r} b^r_{\alpha}\bar{b}^r_{\beta}\,e^{-i(k,x-x')}=-iS^{-}_{\alpha\beta}(x-x')=-\frac{i}{2}(S+iS^{(1)})(x-x').\tag{10}$$

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[1] See eq. (84), (85), (86) and p. 224 below. On the rightside of the eq. (86) l.c. read $+\vec{\alpha}\vec{k}$ instead of $-\vec{\alpha}\vec{k}$ in the expression for $\sum_{k} b^r_{\rho}b^{*r}_{\rho}$.

[2] An asterisk always means the conjugate complex for c-numbers and the hermitian conjugates for operators. The spin indices are dropped if possible without ambiguities.

[3] J. Schwinger, Phys. Rev. **74** (1948), 1939; **75** (1949), 651; **76** (1949), 790. We are using here as much as possible of Schwinger's notations.

For all theories considered we define the energy or Hamiltonian in absence of interaction as

$$E=-\int\bar{\psi}\gamma^4\frac{\partial\psi}{\partial x_4}d^3x=\int\bar{\psi}\left(\vec{\gamma}\frac{\partial}{\partial\vec{x}}+m\right)\psi d^3x, \tag{11a}$$

or also

$$E=+\int\frac{\partial\bar{\psi}}{\partial x_4}\gamma^4\psi d^3x=\int\left(-\frac{\partial\bar{\psi}}{\partial\vec{x}}\vec{\gamma}+m\bar{\psi}\right)\psi d^3x, \tag{11b}$$

which gives according to (4), (5), (6)

$$E=\sum_{k,r}\omega(\hat{u}_r u_r-v_r\hat{v}_r). \tag{12}$$

The canonical commutation laws

$$[E,\ \psi]=-i\frac{\partial\psi}{\partial t},\quad [E,\ \bar{\psi}]=-i\frac{\partial\bar{\psi}}{\partial t} \tag{13}$$

are in view of (4) equivalent to

$$[E,\ u_r]=-\omega u_r,\quad [E,\ v_r]=-\omega v_r, \tag{14a}$$

$$[E,\ \hat{u}_r]=+\omega\hat{u}_r,\quad [E,\ \hat{v}_r]=+\omega\hat{v}_r. \tag{14b}$$

In agreement with the independence of different proper vibration we fulfil (12), (14) by assuming

$$[\hat{u}_r u_r,\ u_s]=-u_r\delta_{rs},\quad [\hat{u}_r u_r,\ \hat{u}_s]=+\hat{u}_r\delta_{rs}, \tag{15a}$$

$$[v_r\hat{v}_r,\ v_s]=+v_r\delta_{rs},\quad [v_r\hat{v}_r,\ \hat{v}_s]=-\hat{v}_r\delta_{rs}, \tag{15b}$$

$$[\hat{u}_r u_r,\ v_s]=[\hat{u}_r u_r,\ \hat{v}_s]=[\hat{v}_r v_r,\ u_s]=[\hat{v}_r v_r,\ \hat{u}_s]=0.$$

To illustrate the properties of charge conjugation we use Schwinger's matrix C defined by[1]

$$\left.\begin{array}{l}\gamma^{\nu T}=\gamma^{\nu *}=-C^{-1}\gamma^{\nu}C\\ CC^{+}=1,\quad C^{T}=-C\end{array}\right\} \tag{16}$$

and choose $b_\rho e^{-i(k,x)}$ as the charge conjugate solution to $a_\rho e^{i(k,x)}$ according to

$$b=C\bar{a},\quad \bar{b}=C^{-1}a. \tag{17}$$

We reserve, however, a change of sign in the usual connection between the charge conjugate field operators $\psi_\alpha'(x)$ with the original adjoint $\bar{\psi}_\alpha(x)$ for the abnormal case to which we shall later return. Here we note the relation

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[1] This matrix is different from the one denoted by C in I. In my earlier paper, Ann. de l'Inst. Poincaré **6** (1936), 109, Schwinger's matrix C is denoted by $B\gamma^5$. Applied to a matrix the notation T means the transposed, $*$ the conjugate complex and $+$ the hermitian conjugate matrix.

$$(C^{-1}S^{+}(-x)C)_{\alpha\beta}=-S^{-}_{\beta\alpha}(x),$$
$$(C^{-1}S^{-}(-x)C)_{\alpha\beta}=-S^{+}_{\beta\alpha}(x), \tag{18}$$

or also

$$(C^{-1}S(-x)C)_{\alpha\beta}=-S_{\beta\alpha}(x), \tag{19}$$

$$(C^{-1}S^{(1)}(-x)C)_{\alpha\beta}=+S^{(1)}_{\beta\alpha}(x), \tag{20}$$

satisfied by Schwinger's S-functions.

A) *The normal case (exclusion principle).*

In order to facilitate the comparison with the abnormal theories we first briefly collect some known formulas for the normal case of spin 1/2-particles obeying the exclusion principle. One assumes first

$$\hat{u}_r=u_r^*,\quad \hat{v}_r=v_r^*, \tag{A 21}$$

and hence

$$\bar{\psi}=\psi^*\gamma^4, \tag{A 22}$$

moreover in agreement with (15a), (15b)

$$\{u_r,\ u_s^*\}=\{v_r,\ v_s^*\}=\delta_{rs}, \tag{A 23}$$

and all other anticommutators between these operators are zero.

Hence:
$$\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}=-iS_{\alpha\beta}(x-x'). \tag{A 24}$$

Dirac's idea of holes is expressed by the assumption that the particle numbers are defined by

$$N_r^+=u_r^*u_r,\qquad \text{hence}\qquad 1-N_r^+=u_ru_r^*, \tag{A 25}$$
$$N_r^-=v_r^*v_r,\qquad \text{,,}\qquad 1-N_r^-=v_rv_r^*,$$

or, in other words, one assumes for the vacuum

$$\langle u_r^*u_r\rangle_0=0,\quad \langle u_ru_r^*\rangle_0=1, \tag{A 26}$$

and

$$\langle v_r^*v_r\rangle_0=0,\quad \langle v_rv_r^*\rangle_0=1.$$

As the energy (12) with help of (A 21), (A 25) takes the form

$$E=\sum_{r,k}\omega\,(N_r^+ + N_r^- -1), \tag{A 27}$$

the used definition of the particle numbers agrees with the condition that the vacuum has to be the state of lowest energy. From (4), (9), (10) and (A 26) one obtains immediately

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0=-iS^{+}_{\alpha\beta}(x-x')\ ;\ \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0=-iS^{-}_{\alpha\beta}(x-x'),$$
$$\langle[\psi_\alpha(x),\ \bar{\psi}_\beta(x')]\rangle_0=-S^{(1)}_{\alpha\beta}(x-x'). \tag{A 28}$$

The anticommutator (A 23) and the vacuum expectation values (A 26) are invariant with respect to charge conjugation if we define the latter by

$$\begin{aligned} u_r' &= v_r, \quad u_r'^{*} = v_r^{*}, \\ v_r' &= u_r, \quad v_r'^{*} = u_r^{*}, \end{aligned} \tag{A 29}$$

which correctly interchanges N^+ and N^- according to (A 25). As this transformation according to (4) and (17) is equivalent to

$$\psi' = C\bar{\psi}, \quad \bar{\psi}' = C^{-1}\psi, \tag{A 30}$$

this latter transformation preserves also the anticommutator (A 24) and the vacuum expectation values (A 28). This can also be proved directly by deriving from (A 30) with help of (16) the relations

$$\begin{aligned} \psi_\alpha'(x)\bar{\psi}_\beta'(x') &= -(C\bar{\psi}(x)\psi(x')C^{-1})_{\alpha\beta}, \\ \bar{\psi}_\beta'(x)\psi_\alpha'(x) &= -(C\psi(x')\bar{\psi}(x)C^{-1})_{\alpha\beta}, \end{aligned} \tag{A 31}$$

and therefore

$$\begin{aligned} \{\psi_\alpha'(x), \bar{\psi}_\beta'(x')\} &= -(C\{\bar{\psi}(x), \psi(x')\}C^{-1})_{\alpha\beta} = -(C^{-1}\{\psi(x'), \bar{\psi}(x)\}C)_{\beta\alpha}, \\ [\psi_\alpha'(x), \bar{\psi}_\beta'(x')] &= -(C[\bar{\psi}(x), \psi(x')]C^{-1})_{\alpha\beta} = +(C^{-1}[\psi(x'), \bar{\psi}(x)]C)_{\beta\alpha}. \end{aligned} \tag{A 32}$$

The comparison of this result with (18), (19), (20) leads directly to the charge invariance of (A 24) and (A 28).

In order to obtain an expression of the current which changes its sign for the transition to the charge conjugate quantities we notice first that in a quantized field theory the two quantities $\psi_\alpha(x)\bar{\psi}_\beta(x')$ and $-\bar{\psi}_\beta(x')\psi_\alpha(x)$ differ only by a c-number. Therefore one can replace the expression of the c-number theory for the current, namely $j_\mu(x) = ie\bar{\psi}(x)\gamma_\mu\psi(x)$ by

$$j_\mu(x) = -\frac{ie}{2}[\psi_\alpha(x), \bar{\psi}_\beta(x)]\gamma^\mu_{\beta\alpha} \tag{A 33}$$

as was first proposed by Heisenberg.1) For the total charge one gets then indeed

$$Q = \int(-i)j_4(x)d^3x = \frac{e}{2}\sum_r\sum_k([u_r^{*}u_r] - [v_r^{*}v_r]) = e\sum_{r,k}(N_r^+ - N_r^-). \tag{A 34}$$

In order to ensure the relation

$$j_\mu'(x) = -j_\mu(x), \tag{A 35}$$

for every space-time point one has, however, to avoid a certain ambiguity in the value of the commutator $[\psi_\alpha(x), \bar{\psi}_\beta(x')]$ in the case of equal space-time points $x = x'$ by defining the limiting process $x' \to x$. Now it follows from (A 32) and (16) that the charge conjugates of $[\psi_\alpha(x), \bar{\psi}_\beta(x')]\gamma^\mu_{\beta\alpha}$ and $[\psi_\alpha(x'), \bar{\psi}_\beta(x)]\gamma^\mu_{\beta\alpha}$ are $-[\psi_\alpha(x'), \bar{\psi}_\beta(x)]\gamma^\mu_{\beta\alpha}$ and $-[\psi_\alpha(x), \bar{\psi}_\beta(x')]\gamma^\mu_{\beta\alpha}$ respectively.

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1) ZS. f. Phys. 90 (1934), 209.

The sum of both expressions has therefore the required property of just changing sign under charge conjugation. If we therefore define

$$[\psi_\alpha(x),\ \bar\psi_\beta(x)] = \lim_{\xi\to0} \tfrac{1}{2}([\psi_\alpha(x+\xi),\ \bar\psi_\beta(x-\xi)] + [\psi_\alpha(x-\xi),\ \bar\psi_\beta(x+\xi)]), \quad \text{(A 46)}$$

we are sure of (A 35) (as this relation then even holds before one goes to the limit $\xi\to0$), which also ensures

$$\langle j_\mu(x)\rangle_0 = 0, \quad \text{(A 37)}$$

in view of the charge invariance of all vacuum-expectation values.

Finally we notice that an introduction of the commutator in the expressions (11) or (12) for the energy (analogous to the expression (A 33) for the charge) does not change the total zero-point-energy and can therefore be omitted.

B) *The abnormal case* (*Bose-statistics*).

In case of Bose-statistics the commutators instead of the anticommutators between field-quantities have to become *c*-numbers and this can be done in accordance with (13)–(15) only by putting

$$[u_r,\ \hat u_s] = \delta_{rs}, \quad [v_r,\ \hat v_s] = -\delta_{rs} \quad \text{(B 23)}$$

and all other commutators between these operators equal to zero. From this it follows in view of (4) that

$$[\psi_\alpha(x),\ \bar\psi_\beta(x')] = -iS_{\alpha\beta}(x-x'). \quad \text{(B 24)}$$

But from now on different assumptions are possible. One way is to retain the connection (A 21), (A 22) between $\bar\psi$, $\hat u$, $\hat v$ and ψ^*, u^*, v^* respectively. This would lead to

$$N_r^+ = \hat u_r u_r, \qquad N_r^+ + 1 = u_r \hat u_r,$$

$$N_r^- = v_r \hat v_r, \qquad N_r^- + 1 = \hat v_r v_r,$$

where the N's have the eigenvalues 0, 1, 2···. The energy given by (12) becomes

$$E = \sum_{k,r} \omega(N_r^+ - N_r^-),$$

while the charge becomes

$$Q = e\int \bar\psi\gamma^4\psi d^3x = e\sum_{k,r} (N_r^+ + N_r^- - 1).$$

This means the existence of particles with negative energy which all have the same sign of charge, so that their number stays always constant, or in other words, there is no pair generation and annihilation. This has the consequence that the state (called '0') in which all particle numbers vanish remains theoretically stable, although it can hardly be considered as 'vacuum' in such a theory due to the existence of the negative energy states.

The expectation values of the bilinear expressions of the field quantities in this 'zero state' are given by

$$\langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0=0,$$

hence

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0=\langle\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}\rangle_0=-iS_{\alpha\beta}(x-x').$$

As all particles have the same charge there would be no charge-conjugation.

There exists, however, the second possibility of changing the connection between the 'adjoint' operators $\bar{\psi}$, $\hat{u}$, $\hat{v}$ and the hermitian conjugates of ψ, u, v. In this case the operators corresponding to physical observables are not any longer hermitian, but 'self-adjoint'.[1] If in particular the Hamiltonian of a system is self-adjoint, the quantity which is conserved with time is not any longer $\sum\limits_n \Psi_n{}^*\Psi_n$ but

$$\sum_n\sum_m \Psi_n{}^*\eta_{nm}\Psi_m=\text{const.}. \tag{B 38}$$

Here the discrete index n is an abbreviating notation for the argument of the Ψ-function in an arbitrary representation and η is an operator which has to fulfil the condition

$$\eta\hat{O}=O^*\eta \tag{B 39}$$

connecting the adjoint $\hat{O}$ with the hermitian-conjugate O^* of every physical operator O of the theory.[2] One gets something new—and as we believe something non-physical—if the quadratic form is not positive definite. In a representation where η is diagonal one can see from (B 38) that (up to a possible normalization factor independent of n) $\eta_n|\Psi_n|^2$ has to be formally interpreted as the "probability" that in the state characterized by Ψ_n the variables of the Ψ-function have a certain set of values characterized by the corresponding specification of the letter "n". If η_n can be negative, "negative probabilities" are formally introduced in this way. Moreover we define as the expectation value $\langle A\rangle$ of the operator A in the state Ψ_n the expression[3]

$$\rangle=\frac{T_r(\Psi^*\eta A\Psi)}{T_r(\Psi^*\eta\Psi)}. \tag{B 40}$$

This definition fails in states where the denominator vanishes (that means where the state vector in the Hilbert space has the length zero), but these states must play a rather singular role in such a theory anyhow.

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[1] If $\bar{\psi}$ is considered as the adjoint to ψ, the imaginary coordinate $x_4=it$ (rather than t), and more generally the components of all vectors, have to be considered as self-adjoint.

[2] See literature quoted above in Note 1.

[3] In II, eq. (3) I have omitted the denominator of (B 40), but the definition (B 40) seems the more natural one because only then the expectation value of a constant simply reproduces the value of this constant.

The theory which we will investigate here is characterized by the particular assumption[1]

$$\hat{u}_r(k)=u_r^*(k),\quad v_r(k)=-v_r^*(k), \tag{B 21}$$

which leads in connection with (B 23) to[2]

$$\begin{aligned} N_r^+&=\hat{u}_r u_r, & N_r^-+1&=u_r\hat{u}_r, \\ N_r^-&=-\hat{v}_r v_r, & N_r^-+1&=-v_r\hat{v}_r, \end{aligned} \tag{B 25}$$

where N^+, N^- have both the eigenvalues 0, 1, 2,⋯.

The condition (B 39) for η here takes the form

$$\eta\hat{v}_r+\hat{v}_r\eta=0,\quad \eta\hat{u}_r-\hat{u}_r\eta=0,$$

which is solved by

$$\eta=(-1)^{\sum\limits_{r,k} N_r^-}\,. \tag{B 41}$$

Hence η is diagonal in a representation where the $N^-(k)$ are diagonal and the time independent quadratic form (B 38) becomes

$$\sum_{(\cdots N_{r,k}^+\cdots;\,\cdots N_{r,k}^-\cdots)}(-1)^{\sum N_r^-}|\Psi(\cdots N_{r,k}^+\cdots;\ \cdots N_{r,k}^-\cdots)|^2=\text{const.} \tag{B 41'}$$

And the expectation value of any function $F(\cdots N_{r,k}^+\cdots;\cdots N_{r,k}^-)$ in this state $\Psi(\cdots N_{r,k}^+\cdots;\cdots N_{r,k}^-\cdots)$ is given by

$$\langle F(\cdots N_{r,k}^+\cdots;\cdots N_{r,k}^-\cdots)\rangle_\Psi=$$

$$=\frac{\sum\limits_{N^+,\,N^-}F(\cdots N^+\cdots;\cdots N^-\cdots)\,(-1)^{\sum\limits_{r,k}N_r^-}\,|\Psi(\cdots N^+\cdots;\cdots N^-\cdots)|^2}{\sum\limits_{N^+,\,N^-}(-1)^{\sum\limits_{r,k}N_r^-}\,|\Psi(N^+\cdots;\cdots N^-\cdots)|^2}\,. \tag{B 42}$$

In the next section we shall derive characteristic consequences of this definition if an external electromagnetic field is present. Inserting now (B 25) in (12) one obtains for the energy

$$E=\sum\omega(N_r^++N_r^-+1) \tag{B 27}$$

with a reversed sign of the zero point energy, but with the state $N^+=N^-=0$ still having the lowest energy thus again playing the role of the vacuum. For the vacuum-expectation values bilinear in $\psi(x)$ and $\bar{\psi}(x')$ one obtains from (B 25) analogous to (A 28)

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[1] Compare II, Section 3.

[2] This is just the type of theory discussed by Dirac in 1942.

$$\langle\psi_\alpha(x)\psi_\beta(x')\rangle_0=-iS^+_{\alpha\beta}(x-x'),\ \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0=+iS^-_{\alpha\beta}(x-x'),$$
$$\langle\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}\rangle_0=-S^{(1)}_{\alpha\beta}(x-x'). \tag{B 28}$$

It is characteristic for this "abnormal" case that the occurrence of the functions $S^{(1)}$ and S in the anticommutator and commutator is just reversed in comparison with the normal case.

The charge conjugation has now to be defined by

$$\begin{aligned} u_r&=v_r, & \hat{u}_r'&=-\hat{v}_r, \\ v_r'&=-u_r, & \hat{v}_r'&=\hat{u}_r \end{aligned} \tag{B 29}$$

in order to preserve the commutation law (B 23) and to interchange N^+ and N^- ($N'^+=N^-$, $N'^-=N^+$). Moreover η is just invariant with respect to this transformation.

According to (4) and (17) this transformation is equivalent to

$$\psi'=C\bar{\psi},\qquad \bar{\psi}'=-C^{-1}\psi. \tag{B 30}$$

The minus sign in the last equation just reverses the sign in the equation analogous to (A 31) and (A 32)

$$\left.\begin{aligned} \psi_\alpha'(x)\bar{\psi}_\beta'(x')&=+(C\bar{\psi}(x)\psi(x')C^{-1})_{\alpha\beta} \\ \bar{\psi}_\beta'(x')\psi_\alpha'(x)&=+(C\psi(x')\bar{\psi}(x)C^{-1})_{\alpha\beta} \end{aligned}\right\}, \tag{B 31}$$

$$\begin{aligned} \{\psi_\alpha'(x),\ \bar{\psi}_\beta'(x')\}&=+(C\{\bar{\psi}(x),\ \psi(x')\}C^{-1})_{\alpha\beta}=+(C^{-1}\{\psi(x'),\ \bar{\psi}(x)\}C)_{\beta\alpha}, \\ [\psi_\alpha'(x),\ \bar{\psi}_\beta'(x')]&=+(C[\bar{\psi}(x),\ \psi(x')]C^{-1})_{\alpha\beta}=-(C^{-1}[\psi(x'),\ \bar{\psi}(x)]C)_{\beta\alpha}. \end{aligned} \tag{B 32}$$

The comparison with (18), (19), (20) lead again to the charge invariances of (B 24) and (B 28) due to the interchange of the role of the S and $S^{(1)}$ function in comparison with the 'normal' theory.

Regarding the definition of the current it is here the anticommutator which differs from the original expression $j_\mu=ie\bar{\psi}\gamma^\mu\psi$ only by a c-number. Therefore we can replace the latter by

$$j_\mu(x)=\frac{ie}{2}\{\psi_\alpha(x),\ \bar{\psi}_\beta(x)\}\gamma^\mu_{\beta\alpha}, \tag{B 33}$$

which gives the total charge

$$\begin{aligned} Q&=\int -ij_4(x)d^3x \\ &=e\sum_{r,k}(\{\hat{u}_r,\ u_r\}+\{\hat{v}_r,\ v_r\}) \\ &=e\sum(N_r^+-N_r^-). \end{aligned} \tag{B 34}$$

Moreover the charge conjugates of $\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}\gamma^\mu_{\alpha\beta}$ and $\{\psi_\alpha(x'),\ \bar{\psi}_\beta(x)\}\gamma^\mu_{\beta\alpha}$ are $-\{\psi_\alpha(x'),\ \bar{\psi}_\beta(x)\}\gamma^\mu_{\beta\alpha}$ and $-\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}\gamma^\mu_{\beta\alpha}$ respectively, the sum of both terms having therefore the required property of changing sign under charge conjugation. We define therefore

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x)\}=\lim_{\xi\to 0}\tfrac{1}{2}\left(\{\psi_\alpha(x+\xi),\bar{\psi}_\beta(x-\xi)\}+\{\psi_\alpha(x-\xi),\bar{\psi}_\beta(x-\xi)\}\right) \qquad \text{(B 36)}$$

to ensure

$$j_\mu'(x)=-j_\mu(x), \qquad \text{(B 35)}$$

and

$$\langle j_\mu(x)\rangle_0=0. \qquad \text{(B 37)}$$

Just as the 'normal' theory which applies the exclusion principle this 'abnormal' theory of Bosons with spin 1/2, but with 'negative probabilities' gives pair generation and annihilation in an external electromagnetic field as not the particle number but the charge which is capable of both signs is here conserved.

§ 3. Spin 1/2 particles in an external electromagnetic field.

We consider now the presence of an external non quantized electromagnetic field with the potential $A_\mu(x)$ and the interaction energy

$$H_{int}=-j_\mu(x)A_\mu(x) \qquad (43)$$

with the quantized matter field. The switching in of the electromagnetic field ($A_\mu=0$ for $t\to-\infty$) gives rise to a change of the Schrödinger wave function (in the interaction representation) given by

$$\Psi(t)=U(t,\ -\infty)\Psi(-\infty). \qquad (44)$$

As shown by Dyson, $U(t,\ -\infty)$ can be written in the form

$$U(t,\ -\infty)=P\left(e^{i\int_{-\infty}^{t} j_\mu(x')A_\mu(x')d^4x'}\right) \qquad (45)$$

$$=\sum_0^\infty\frac{i^n}{n!}\int_{-\infty}^{t}d^4x_1\cdots\int_{-\infty}^{t}d^4x_n P(j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n))A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n),$$

where P is an operator which establishes the chronological order of factors in such a way that the time values decrease from left to right. One obtains the reciprocal of U by changing the sign of i and reversing the chronological order. If the interaction energy is Hermitian, U is unitary, if the interaction energy, however, is only "self-adjoint" one has instead of it

$$U\hat{U}=1, \qquad (46)$$

where $\hat{U}$ is the adjoint of U.

We shall be particularly interested in the probability that the vacuum is preserved for $t=\infty$ which is given by

$$U_0=|\langle U(\infty,\ -\infty)\rangle_0|^2=\langle P(e^{i\int_{-\infty}^{+\infty}j_\mu(x')A_\mu(x')d^4x'})\rangle_0\langle P_-(e^{-i\int_{-\infty}^{+\infty}j_\mu(x')A_\mu(x')d^4x'})\rangle_0, \tag{47}$$

where P_- is the operator which establishes the reverse order of time. This holds also for the abnormal case as for the vacuum the sign in (B 41) is positive. In the following there is no essential restriction as long as the potentials are arbitrary functions of time.

To the second order in e one obtains

$$\langle(U(-\infty,\ \infty)\rangle_0=1+i\int_{-\infty}^{+\infty}\langle j_\mu(x)\rangle_0 A_\mu(x)d^4x-$$

$$-\frac{1}{2}\int_{-\infty}^{+\infty}d^4x\int_{-\infty}^{+\infty}d^4x' A_\mu(x)A_\nu(x')\langle P(j_\mu(x),\ j_\nu(x'))\rangle_0.$$

Using

$$\langle j_\mu(x)\rangle_0=0,$$

$$P(j_\mu(x),\ j_\nu(x'))=\frac{1}{2}\{j_\mu(x),\ j_\nu(x')\}+\frac{1}{2}\epsilon(x-x')[j_\mu(x),\ j_\nu(x')],$$

$$P_-(j_\mu(x),j_\nu(x'))=\frac{1}{2}\{j_\mu(x),\ j_\nu(x)\}-\frac{1}{2}\epsilon(x-x')[j_\mu(x),\ j_\nu(x')],$$

one gets

$$W_0=|\langle U(-\infty,\ \infty)\rangle_0|^2=1-\frac{1}{2}\int_{-\infty}^{+\infty}d^4x\int_{-\infty}^{+\infty}d^4x' A_\mu(x)A_\nu(x')\langle\{j_\mu(x),\ j_\nu(x')\}\rangle_0. \tag{48}$$

We shall now check Feynman's result that for the abnormal case the second term has a sign opposite to that of the normal case. As in the latter the sign of this term is negative, one obtains a *probability larger than 1* for the realisation of the vacuum in the abnormal case. This, however, is just to be expected from the general theorem (B 41′). In the approximation in question besides the vacuum ψ only the terms corresponding to states where one pair is generated contribute to the sum of its left side. In these states one of the N^+ and one of the N^- is equal 1 so that the sign is negative. Although one still has a conservation of "probability" one gets here $W_0>1$ due to the non-unitarity of the matrix U, or in other words due to the negative sign of the probability of some states.

One has indeed

$$\langle j_\mu(x)j_\nu(x')\rangle_0=-e^2\gamma^{\mu}_{\alpha\beta}\langle\psi_\beta(x)\bar{\psi}_\rho(x')\rangle_0\gamma^{\nu}_{\rho\sigma}\langle\bar{\psi}_\alpha(x)\psi_\sigma(x')\rangle_0.$$

Therefore in the normal case according to (A 28)

$$\langle j_\mu(x)j_\nu(x')\rangle_0=e^2T_r(\gamma^\mu S^+(x-x')\gamma^\nu S^-(x'-x)). \tag{A 48}$$

And in the abnormal case according to (B 28)

$$\langle j_\mu(x) j_\nu(x')\rangle_0 = -e^2 T_r(\gamma^\mu S^+(x-x')\gamma^\nu S^-(x'-x)). \tag{B 48}$$

One also obtains for the normal case

$$\begin{aligned}\langle\{ j_\mu(x), j_\nu(x')\}\rangle_0 &= e^2 T_r(\gamma^\mu S^+(x-x')\gamma^\nu S^-(x'-x) + \gamma^\nu S^+(x'-x)\gamma^\mu S^-(x-x')) \\ &= e^2 T_r(\gamma^\mu S^+(x-x')\gamma^\nu S^-(x'-x) + \gamma^\mu S^-(x-x')\gamma^\nu S^+(x'-x)),\end{aligned}$$

or also

$$\langle\{ j_\mu(x), j_\nu(x')\}\rangle_0 = \frac{1}{2} e^2 T_r(\gamma^\mu S^{(1)}(x-x')\gamma^\nu S^{(1)}(x'-x) + \gamma^\mu S(x-x')\gamma^\nu S(x'-x)), \tag{A 49}$$

and for the abnormal case the opposite

$$\langle\{ j_\mu(x), j_\nu(x')\}\rangle_0 = -\frac{1}{2} e^2 T_r(\gamma^\mu S^{(1)}(x-x')\gamma^\nu S^{(1)}(x'-x) + \gamma^\mu S(x-x')\gamma^\nu S(x'-x)). \tag{B 49}$$

It is easy to evaluate the traces but for our purpose, to compare the consequences of the present formalism with Feynman's result, the confirmation is sufficient that the second term in W_0 has opposite signs for the abnormal and normal cases.

§ 4. Spin O-particles (scalar fields).

We assume a scalar field $\phi(x)$ and its adjoint field $\hat{\phi}(x)$ both fulfilling the second order wave equation[1]

$$(\Box - m^2)\phi(x) = 0, \quad (\Box - m^2)\hat{\phi}(x) = 0 \tag{50}$$

in absence of any interaction, but leaving open the connection between $\hat{\phi}$ and the hermitian conjugate ϕ^* of ϕ.

The decomposition of $\phi(x)$ into plane waves we write as

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega}} \{u(k) e^{i(kx)} + \hat{v}(k) e^{-i(k,x)}\}, \tag{51}$$

$$\hat{\phi}(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega}} \{\hat{u}(k) e^{-i(k,x)} + v(k) e^{+i(k,x)}\}, \tag{52}$$

where again $k_4 = i\omega = i\sqrt{\vec{k}^2 + m^2}$. One has

$$\begin{aligned}\frac{1}{V}\sum_k \frac{1}{2\omega} e^{i(k,\, x-x')} &= \left(\frac{1}{2\pi}\right)^3 \int \frac{d^3k}{2\omega} e^{i(k,x-x')} = \\ &= +i\Delta^+(x-x') = +\frac{i}{2}(\Delta - i\Delta^{(1)})(x-x'),\end{aligned} \tag{53}$$

---

[1] We denote the D'Alembertian operator $\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}$ with $\Box$ and not as Schwinger with $\Box^2$.

$$\begin{aligned}\frac{1}{V}\sum_k \frac{1}{2\omega}e^{-i(k,x-x')} &= \left(\frac{1}{2\pi}\right)^3\int\frac{d^3k}{2\omega}e^{-i(k,x-x')}\\ &= -i\Delta^-(x-x')\\ &= -\frac{i}{2}(\Delta+i\Delta^{(1)})(x-x'). \qquad (54)\end{aligned}$$

We also note the well known relations

$$\Delta(-x)=-\Delta(x)\,;\ \Delta^{(1)}(-x)=\Delta^{(1)}(x),$$
$$-\Delta^+(-x)=\Delta^-(x)=(\Delta^+(x))^*\,;\ -\Delta^-(-x)=\Delta^+(x)=(\Delta^-(x))^*, \qquad (55)$$

and the fact that $\Delta(x)$ and $\Delta^{(1)}(x)$ are real. The expression for the energy is here

$$E=\int d^3x\left(\frac{\partial\hat{\phi}}{\partial t}\frac{\partial\phi}{\partial t}+\frac{\partial\hat{\phi}}{\partial\vec{x}}\frac{\partial\phi}{\partial\vec{x}}+m^2\hat{\phi}\phi\right), \qquad (56)$$

which geos with help of (51), (52) into

$$E=\sum_k\omega(\hat{u}u+v\hat{v}). \qquad (57)$$

The canonical commutation laws

$$[E,\ \phi]=-i\frac{\partial\phi}{\partial t}\,;\ [E,\ \hat{\phi}]=-i\frac{\partial\hat{\phi}}{\partial t} \qquad (58)$$

are in analogy with (14a), (14b) equivalent to

$$[E,\ u]=-\omega u,\quad [E,\ v]=-\omega v, \qquad (59a)$$
$$[E,\ \hat{u}]=+\omega\hat{u},\quad [E,\ \hat{v}]=+\omega\hat{v}, \qquad (59b)$$

which are fulfilled in agreement with the independence of different proper vibrations if

$$[\hat{u}_k u_k,\ u_{k'}]=-u_k\delta_{k'k}\,;\ [\hat{u}_k u_k,\ \hat{u}_{k'}]=+\hat{u}_k\delta_{k'k}, \qquad (60a)$$
$$[v_k\hat{v}_k,\ v_{k'}]=-v_k\delta_{k'k}\,;\ [v_k\hat{v}_k,\ \hat{v}_{k'}]=+\hat{v}_k\delta_{kk'}, \qquad (60b)$$
$$[\hat{u}_k u_k,\ v_{k'}]=[\hat{u}_k u_k,\ \hat{v}_{k'}]=[\hat{v}_k v_k,\ u_{k'}]=[\hat{v}_k v_k,\ \hat{u}_{k'}]=0.$$

A) *The normal case* (*Bose statistics*).

One assumes

$$\hat{u}=u^*,\quad \hat{v}=v^*, \qquad (\text{A } 61)$$

therefore

$$\hat{\phi}(x)=\phi^*(x). \qquad (\text{A } 62)$$

Moreover in agreement with (60a), (60b)

$$[u_k,\ u^*_{k'}]=[v_k,\ v^*_{k'}]=\delta_{kk'}. \qquad (\text{A } 63)$$

Hence $$[\phi(x),\ \phi^*(x')]=[\phi^*(x),\ \phi(x')]=i\Delta(x-x'). \qquad (\text{A } 64)$$

The commutation rules (A 63) have the consequence

$$\begin{aligned} N_k^+&=u_k^*u_k, & N_k^+ +1&=u_ku_k^*, \\ N_k^-&=v_k^*v_k, & N_k^- +1&=v_kv_k^*, \end{aligned} \qquad (\text{A } 65)$$

where the N have the eigenvalues 0, 1, 2⋯which makes the energy (57)

$$E=\sum\omega(N_k^+ + N_k^- +1), \qquad (\text{A } 66)$$

so that the vacuum $N^+=N^-=0$ has the lowest energy. For this state one gets

$$\left.\begin{aligned} \langle u_k^*u_k\rangle_0&=0, & \langle u_ku_k^*\rangle_0&=1 \\ \langle v_k^*v_k\rangle_0&=0, & \langle v_kv_k^*\rangle_0&=1 \end{aligned}\right\}, \qquad (\text{A } 67)$$

$$\left.\begin{aligned} \langle\phi(x)\phi^*(x')\rangle_0&=\langle\phi^*(x)\phi(x')\rangle_0=+i\Delta^+(x-x') \\ \langle\phi^*(x')\phi(x)\rangle_0&=\langle\phi(x')\phi^*(x)\rangle_0=-i\Delta^-(x-x') \end{aligned}\right\}, \qquad (\text{A } 68)$$

$$\langle\{\phi(x),\ \phi^*(x')\}\rangle_0=+\Delta^{(1)}(x-x'). \qquad (\text{A } 69)$$

The commutator (A 64) and the vacuum expectation values (A 67), (A 68), (A 69) are invariant with respect to charge-conjugation

$$\begin{aligned} u'&=v, & u'^*&=v^*, \\ v'&=u, & v'^*&=u^*, \end{aligned} \qquad (\text{A } 70)$$

which gives correctly $N^{+\prime}=N^-$, $N^{-\prime}=N^+$. According to (51), (52) this is simply equivalent to

$$\phi'=\phi^*, \qquad \phi'^*=\phi. \qquad (\text{A } 71)$$

For the current we use the symmetrized expression

$$j_\mu(x)=\frac{ie}{2}\left(\left\{\frac{\partial\phi^*(x)}{\partial x_\mu},\phi(x)\right\}-\left\{\phi^*(x),\frac{\partial\phi(x)}{\partial x_\mu}\right\}\right), \qquad (\text{A } 72)$$

which leads to the correct total charge

$$Q=\int(-i)j_4(x)d^3x=\frac{e}{2}\sum(\{u_k,u_k^*\}-\{v_k,v_k^*\})=e\sum(N_k^+-N_k^-). \quad (\text{A } 73)$$

In order to ensure the relation

$$j_\mu'(x)=-j_\mu(x), \qquad (\text{A } 74)$$

one has here (in analogy with the corresponding commutator for spinor fields) to define the anticommutator, as

$$\left\{\frac{\partial\phi^*(x)}{\partial x_\mu},\phi(x)\right\}=\lim_{\xi=0}\frac{1}{2}\left(\left\{\frac{\partial\phi^*(x+\xi)}{\partial x_\mu},\phi(x-\xi)\right\}+\right.$$
$$\left.+\left\{\frac{\partial\phi^*(x-\xi)}{\partial x_\mu},\phi(x+\xi)\right\}\right),$$

$$\left\{\frac{\partial\phi(x)}{\partial x_\mu}, \phi^*(x)\right\} = \lim_{\xi\to 0}\frac{1}{2}\left(\left\{\frac{\partial\phi(x+\xi)}{\partial x_\mu},\ \phi^*(x-\xi)\right\}+\right.$$
$$\left.+\left\{\frac{\partial\phi(x-\xi)}{\partial x_\mu},\ \phi^*(x+\xi)\right\}\right), \tag{A 75}$$

which also guarantees

$$\langle j_\mu(x)\rangle_0=0. \tag{A 76}$$

We also note vacuum-expectation value of expressions bilinear in the current, which occurred in (48) for this case:

$$\langle j_\mu(x)\ j_\nu(x')\rangle_0=e^2\Big(-\langle\frac{\partial\phi^*(x)}{\partial x_\mu}\phi(x')\rangle_0\langle\phi(x)\frac{\partial\phi^*(x')}{\partial x'_\nu}\rangle_0$$
$$+\langle\frac{\partial\phi^*(x)}{\partial x_\mu}\frac{\partial\phi(x')}{\partial x_\nu'}\rangle_0\langle\phi(x)\phi^*(x')\rangle_0$$
$$-\langle\frac{\partial\phi(x)}{\partial x_\mu}\phi^*(x')\rangle_0\langle\phi^*(x)\frac{\partial\phi(x')}{\partial x_\nu'}\rangle_0$$
$$+\langle\frac{\partial\phi(x)}{\partial x_\mu}\frac{\partial\phi^*(x')}{\partial x_\nu'}\rangle_0\langle\phi^*(x)\phi(x')\rangle_0\Big), \tag{A 77}$$

which gives with help of (A 69)

$$\langle j_\mu(x)j_\nu(x')\rangle_0= +2e^2\Big(-\frac{\partial\Delta^+(x-x')}{\partial x_\mu}\frac{\partial\Delta^+(x-x')}{\partial x_\nu}+$$
$$+\frac{\partial^2\Delta^+(x-x')}{\partial x_\mu\partial x_\nu}\Delta^+(x-x')\Big), \tag{A 78}$$

and also

$$\langle\{j_\mu(x), j_\nu(x')\}\rangle_0=e^2\Big(-\frac{\partial\Delta(x-x')}{\partial x_\mu}\frac{\partial\Delta(x-x')}{\partial x_\nu}+$$
$$+\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\mu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\nu}+\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu}\Delta(x-x')-$$
$$-\frac{\partial^2\Delta^{(1)}(x-x')}{\partial x_\mu\partial x_\nu}\Delta^{(1)}(x-x')\Big). \tag{A 79}$$

B) *The abnormal case* (*Exclusion principle*).

The anticommutators between u, $\hat{u}$ and v, $\hat{v}$ can only be fixed in agreement with (60a) (60b) if one puts

$$\{u_k,\ \hat{u}_{k'}\}=\delta_{kk'}\,;\quad \{v_k,\ \hat{v}_{k'}\}=-\delta_{kk'} \tag{B 63}$$

with all other anticommutators between these quantities equal to zero.[1] If one wishes to retain the identification of $\hat{u}$, $\hat{v}$ with the hermitian conjugates u^*, v^* one has therefore to abandon the energy expression (56) in order to fulfil the canonical commutation relations (58). One obtains then a theory where the function $\triangle^{(1)}$ instead of $\triangle$ occurs in the commutators of the field quantities. I discussed this possibility at an earlier occasion[1] and rejected it for the reason that physical observables have to commute in points with space-like distances. Moreover it turns out to be impossible in this way to obtain the correct field equations with an external electromagnetic field in agreement with canonical commutation rules.

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[1] Compare I, p. 211.

We therefore discuss here the other possibility, to maintain the energy expression (56) but to change the connection of adjoint and hermitian conjugate quantities in a way analogous to the one we have used for spin ½ particles with Bose-statistics, namely

$$\hat{u}(k)=u^*(k), \quad \hat{v}(k)=-v^*(k). \tag{B 21}$$

From (B 63) it follows then that

$$N^+=\hat{u}u, \qquad 1-N^+=u\hat{u},$$
$$N^-=-\hat{v}v, \quad 1-N^-=-v\hat{v}, \tag{B 64}$$

where the N's have the eigenvalues 0 and 1. The expressions (B 41), (B 41'), (B 42) for χ the conserved sum of all (partly negative) probabilities and the expectation value of the operator $F(\cdots N^+\cdots, \cdots N^-\cdots)$ are still valid, except that the sums over the N's are running now over 0 and 1 only. One obtains first

$$\{\phi(x), \hat{\phi}(x')\}=\{\hat{\phi}(x), \phi(x')\}=i\triangle(x-x'). \tag{B 65}$$

The energy expression for the normal case is according to (57) and (B 64)

$$E=\sum_k \omega(N_k^+ + N_k^- - 1) \tag{B 66}$$

with a reversal of the sign of the zero point energy but with $N^+=N^-=0$ for the state of lowest energy, which therefore has to be defined as vacuum.

For the vacuum expectation values of bilinear expressions of field variables one gets now

$$\langle\phi(x)\phi(x')\rangle_0=-\langle\hat{\phi}(x)\phi(x')\rangle_0=i\triangle^+(x-x'),$$
$$\langle\hat{\phi}(x')\phi(x)\rangle_0=-\langle\phi(x')\hat{\phi}(x)\rangle_0=i\triangle^-(x-x'), \tag{B 69a}$$

and also

$$\langle[\phi(x), \hat{\phi}(x')]\rangle_0=\triangle^{(1)}(x-x'). \tag{B 69b}$$

Analogous to (B 29), the charge conjugation

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[1] Ann. de l'Inst. Poincaré **6** (1936), 137.

$$u_k'=v_k, \qquad \hat{u}_k'=-\hat{v}_k,$$
$$v_k'=-u_k, \qquad \hat{v}_k=\hat{u}_k \tag{B 70}$$

leaves the anticommutators and vacuum-expectation-values as well as η invariant and is equivalent to

$$\phi'(x)=\hat{\phi}(x), \quad \hat{\phi}'(x)=-\phi(x). \tag{B 71}$$

The expression for the currents

$$j_\mu(x)=\frac{ie}{2}\left(\left[\frac{\partial\hat{\phi}(x)}{\partial x_\mu}, \phi(x)\right]-\left[\hat{\phi}(x), \frac{\partial\phi(x)}{\partial x_\mu}\right]\right) \tag{B 72}$$

fulfils the conditions (B 74)

$$j_\mu'(x)=-j_\mu(x), \quad \langle j_\mu(x)\rangle_0=0 \tag{B 76}$$

if the brackets are defined with a symmetrizing limit analogous to (A 75), and the value of the charge becomes

$$Q=\int(-i)j_4 d^3x=\frac{e}{2}\sum_k([\hat{u}, u]-[v, \hat{v}])=e\sum_k(N^+-N^-). \tag{B 73}$$

For the expectation values of the expressions bilinear in the currents one gets the same formula as (A 77) but with $\hat{\phi}$ substituted for ϕ^*. But according to (B 69a) and (A 69) in every term one of the factors (the one where $\hat{\phi}$ is in front of ϕ) changes its sign so that again as for spin 1/2

$$(\langle j_\mu(x')j_\nu(x)\rangle_0)_{\text{abnormal}}=-(\langle j_\mu(x)j_\nu(x')\rangle_0)_{\text{normal}}. \tag{B 78, 79}$$

A consideration similar to the one given in § 3 is also possible for the probability here that the vacuum is conserved in an external electromagnetic field. One gets again a value greater than one for this probability in the abnormal case in the same way as for spin 1/2-particles.

I am indebted to Dr. R. Jost for interesting discussions on this subject during my stay as visitor at the Institute for Advanced Study in Princeton.

Progress of Theoretical Physics Vol. 5, No. 4, July~August, 1950

Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems

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(Received June 29, 1950)

Abstract

The fact implied by Bloch several years ago that in some approximate sense the behavior of an assembly of Fermi particles can be described by a quantized field of sound waves in the Fermi gas, where the sound field obeys Bose statistics, is proved in the one-dimensional case. This fact provides us with a new possibility of treating an assembly of Fermi particles in terms of the equivalent assembly of Bose particles, namely, the assembly of sound quanta. The field equation for the sound wave is found to be linear irrespective of the absence or presence of mutual interaction between particles, so that this method is a very useful means of dealing with many-Fermion problems. It is also applicable to the case where the interparticle force is not weak. In the case of force of too short a range this method fails.

§ 1. Introduction and summary.

The well-known method of Thomas and Fermi provides us with a very practical approximate treatment of many-Fermion problems. Because in this approximation, however, each particle is supposed to move independently, the effect of interaction between particles being simply replaced by an average field of force, one cannot speak of correlations between particles in this rough approximation. This simple method does not apply to problems in which inter-particle correlation plays an important role. A step toward the improvement of the method so as to include the correlation was taken by Euler[1], who calculated the effect of inter-particle interaction, which causes the correlation, by perturbation theory assuming the interaction to be small.

The calculation of such a type is carried out in the following manner: In the zeroth approximation each particle is in some one-particle quantum state, which we shall call a " level ". Let us consider the lowest state for the sake of definitness. In this state all levels up to some highest one, which shall be called the " Fermi maximum ", are each filled by one particle. The perturbation energy, the energy due to the inter-particle force, has non-vanishing matrix elements which cause virtual transitions to states in which two particles are excited simultaneously to levels higher than the Fermi maximum.

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The state which results from the inter-particle force is thus such one which is a superposition of the zero-order state and various excited states in which holes and particles are present in some levels below and above the Fermi maximum respectively. In the lowest approximation of the perturbation calculation, the numbers of holes and excited particles are two ; but, if the calculation is carried out to higher order, which is necessary when the inter-particle force is not small, there appear states in which a considerable number of holes and excited particles are present. The amplitudes of such highly excited states will be appreciable if the inter-particle force is strong.

Such a mixture of excited states gives rise to correlation between particles. We can in this way include the effects of correlations in treating certain problems. In the case of strong inter-particle forces, however, it is necessary to carry through the perturbation calculation up to a very high order, and this is too involved to be practicable. Recently, Nogami[2] proposed a method which applies also to cases of rather strong inter-particle forces. Still his method requires some kind of weakness of interaction because he had to neglect the interactions of holes and of excited particles with each other as well as the interactions between holes and exited particles. This neglect cannot be justified when too many holes and excited particles are present.

In such a situation it is desirable to find some approximate method of dealing with many-Fermion problems different from the perturbation method.

In his famous work on the stopping power of charged particles, Bloch[3] has treated the excited states of the Fermi gas not as states with holes and excited particles, but as states in which the gas oscillates. In this work it was not necessary to treat the oscillation quantum-theoretically. In a later paper[4], he also dealt with a problem in which the quantum aspect of the oscillation was essential. This was a problem in which the density fluctuation played a role. He showed in this work that the density fluctuation of a degenerate Fermi gas (in his theory it was sufficient to treat the gas without inter-particle force) can be calculated in two different ways giving the same result. The one was the orthodox method whereby one calculates directly the expectation value of the operator having expressed in terms of the quantized field variables ψ and ψ^* describing the assembly of the Fermi particles. The other method was to calculate the zero-point amplitudes of sound waves in the gas, the equations of motion for the sound waves being properly chosen. He showed that the correct value of the fluctuation was obtained in this way if the choice of the equations of motion for the sound waves was properly made and if the zero-point amplitudes of the waves had such values as would be expected for a sound field obeying Bose statistics.

If it is proved that the excited states of an assembly of Fermi particles can be in fact described as an excitation of sound waves, where the sound can be described by a Bose field, it will provide us with a new method of treating many-Fermion problems, not dealing directly with the assembly of the Fermi particles, but dealing with the equivalent assembly of Bose particles, i. e. the assembly of sound quanta.

There is a prospect that in this latter method the assumption of the weakness of inter-pariicle forces will not be essential. This prospect lies in the following situation: It is expected that the equations of motion for the sound waves will be linear in the field variables describing the sound field; otherwise we could not speak of waves at all. The field variables for the sound field will be the density ρ of the gas and its properly defined canonical conjugate. Now the essential point is that the linearity of the problem will not be destroyed even when an inter-particle force is present, because the interaction energy between particles is bilinear in ρ. This fact is in marked contrast to the circumstance that in terms of ψ and ψ^* the interaction energy contains four ψ's, or, more precisely, two ψ's and twe ψ^*'s; thus the field equations for ψ and ψ^* are no longer linear when an inter-particle force is present. This circumstance made it very difficult to treat many-Fermion problems with inter-particle forces. This difficulty will disappear when we deal with the problem in terms of the ρ field but not of the ψ field.

The purpose of this paper is to show that these expectations are really fulfilled. It will be shown that in some approximation, which does not necessarily require the weakness of the interaction, excitated states of Fermi gas are in fact equivalent to corresponding excitations of sound waves, and that the sound is describable by a Bose field whose field equations are linear in the field variables irrespective of the absence or presence of interparticle forces. Thus, Bloch's method of sound waves will be a very useful method for many-Fermion problems.

The possibility of this new method was found independently by Bohm[5], who discusses a very interesting phenomenon of plasma-like oscillations in a degenerate electron gas from a very similar point of view. He, too, gives a proof of the fact that these oscillations are describable by a Bose field in some approximation. He utilizes further the linearity of the field equations to study such a pure correlation phenomenon as plasma oscillations of an electron gas.

The present paper is of a rather more mathematical nature than physical in that it aims mainly at the analysis of the mathematical structure of the method, clarifying the underlying assumptions and the limit of applicability. A mathematically closed and clear-cut presentation of the theory is achieved, however, at the expense of physical usefulness, because, thus far, the author has succeded only in giving a complete formulation for a one-dimensional assembly of particles.

It is rather certain that a similar method applies to the three dimensional case too—this, indeed, has been done by Bohm but the situation is more complicated in this case.

The mathematicl relation between the field of sound quanta on the one hand and the original field of Fermi particles on the other is very similar to the relation between the field of light quanta and the field of neutrinos in the neutrino theory of light[6]. One will find everywhere a marked parallelism between our theory and the neutrino theory of light.

The discussions will be performed in several steps: In § 2 we shall prove that the sound can be described by a Bose field under some assumptions imposed on the states under consideration. Then we shall set up in § 3 the Hamiltonian for the sound field. We shall see that the Hamiltonian is in fact bilinear in the field variables, so that the whole problem is liner. In § 4 we shall show that this Hamiltonian is really equal to the original Hamiltonian for the assembly of Fermi particles, and, therefore, the assembly of sound quanta is equivalent to the original assembly of Fermi particles. In § 5 we will go over to the solution of the eigenvalue problem. This can be done very simply by finding the normal coordinates for the sound field, because the field equations are linear. In § 6 we shall give several general formulae of physical interest which are derived directly by our method. A criterion for the applicability of the method will also be given in this section. In the last section we shall briefly mention the bearing of our results on the plasma oscillations treated by Bohm. Also, the relation between the two kinds of descriptions of the system, one as an assembly of Fermi particles and the other as an assembly of sound quanta, is discussed briefly.

In § 6 we shall see that our method does not work if the inter-particle force is of short range. The range of force must be larger than four times the mean distance between particles. Since sound waves with wave length shorter than the mean distance between particles have no meaning, it is quite conceivable that the method of sound waves fails in describing the event occuring in a small space region comparable with the mean distance between particles. This is the reason why the method fails in the case of short range force. The method is, on the other hand, very suitable for dealing with the case of long-range force in which a considerable number of holes and excited particles are present in the neighborhood of the Fermi maximum. Thus the present method covers a field where the known methods have failed.

§ 2. Approximate commutation relations for the density field.

Let us consider an assembly of Fermi particles in a one-dimensional " box " of length L. Let $\psi(x)$ and $\psi^*(x)$ be the quantized wave functions describing the assembly. The density $\rho(x)$ of the particles is then given by

$$\rho(x) = \psi^*(x)\psi(x). \tag{2.1}$$

We introduce the Fourier transforms ψ_n and $\psi_n{}^*$ of $\psi(x)$ and $\psi^*(x)$ respectively. They are defined by

$$\begin{cases} \psi(x)=\dfrac{1}{\sqrt{L}}\sum\limits_{n}\psi_n \exp\left(\dfrac{2\pi i}{L}nx\right), \\ \psi^*(x)=\dfrac{1}{\sqrt{L}}\sum\limits_{n}\psi_n{}^* \exp\left(-\dfrac{2\pi i}{L}nx\right), \end{cases} \qquad n=0,\ \pm 1,\ \pm 2,\cdots. \tag{2.2}$$

The Fourier transform of $\rho(x)$, which is defined by

$$\rho(x)=\frac{1}{L}\sum_{n}\rho_n \exp\left(\frac{2\pi i}{L}nx\right), \tag{2.3}$$

is evidently given by

$$\rho_n=\sum_{n'}\psi^*_{n'}\psi_{n'+n}=\sum_{\bar{n}}\psi^*_{\bar{n}-\frac{n}{2}}\psi_{\bar{n}+\frac{n}{2}}, \tag{2.4}$$

where $\bar{n}$ takes integral values when n is even and half-odd integral values when n is odd.

Our purpose is to describe the system in terms of the density field $\rho(x)$ instead of describing it by the wave field $\psi(x)$. The first task is to find the commutation relations between the field quantity $\rho(x)$ and its properly defined canonical conjugate. For this purpose we separate each ρ into two parts ρ^+_n and ρ^-_n by means of

$$\begin{cases} \rho^+_n=\sum\limits_{\bar{n}>0}\psi^*_{\bar{n}-\frac{n}{2}}\psi_{\bar{n}+\frac{n}{2}}, \\ \rho^-_n=\sum\limits_{\bar{n}<0}\psi^*_{\bar{n}-\frac{n}{2}}\psi_{\bar{n}+\frac{n}{2}}. \end{cases} \tag{2.5}$$

We have evidently

$$\rho_n=\rho^+_n+\rho^-_n \tag{2.6}$$

(in case of even n the term with $\bar{n}=0$ is absent in (2.6), but this does not cause any serious error.) We have further

$$\rho^+_n=\rho^{+*}_{-n}, \qquad \rho^-_n=\rho^{-*}_{-n}. \tag{2.7}$$

It will be seen later that the separation of ρ_n by (2.6) corresponds to the separation of the field into parts with positive and negative frequencies, which is the usual procedure in field theory. (See (3.1) and (3.2)).

We now examine the commutation relations of the ρ's. The commutation relations between the ψ's and ψ^*'s are

$$\begin{cases} [\psi_n{}^*,\ \psi_{n'}]_+=\delta_{n,n'}, \\ [\psi_n{}^*,\ \psi_{n'}{}^*]_+=[\psi_n,\ \psi_{n'}]_+=0. \end{cases} \tag{2.8}$$

From these relations we get the commutation relations of the ρ's. For the ρ^+'s they are

$$[\rho^+_n,\ \rho^+_{n'}]=\begin{cases} \sum\limits_{-\frac{n}{2}<\bar{n}\leqq\frac{|n'|}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ n'\leqq -n, \\ \sum\limits_{\frac{n'}{2}<\bar{n}\leqq\frac{n}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ -n\leqq n\leqq n, \\ -\sum\limits_{\frac{n}{2}<\bar{n}\leqq\frac{n'}{2}}\psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n>0,\ n'\geqq n, \end{cases} \tag{2.9}$$

which, for the special cases of $n' = -n$, reduce to

$$[\rho_n^+,\ \rho_{-n}^+] = \sum_{-\frac{n}{2} < \bar{n} \leqq \frac{n}{2}} \psi_{\bar{n}}{}^* \psi_{\bar{n}} \qquad \text{for } n > 0. \tag{2.9'}$$

In the same way, we get for the ρ^-'s

$$[\rho_n^-,\ \rho_{n'}^-] = \begin{cases} -\sum\limits_{\frac{n'}{2} \leqq \bar{n} < \frac{n}{2}} \psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}} \psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n > 0,\ n' \geqq -n, \\ -\sum\limits_{-\frac{n}{2} \leqq \bar{n} < -\frac{n'}{2}} \psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}} \psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n > 0,\ -n \leqq n' \leqq n, \\ \sum\limits_{-\frac{n'}{2} \leqq \bar{n} < -\frac{n}{2}} \psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}} \psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n > 0,\ n' \geqq n, \end{cases} \tag{2.10}$$

and

$$[\rho_n^-,\ \rho_{-n}^-] = -\sum_{-\frac{n}{2} \leqq \bar{n} < \frac{n}{2}} \psi_{\bar{n}}{}^* \ \psi_{\bar{n}} \qquad n > 0. \tag{2.10'}$$

The commutation relations between the ρ^+'s and ρ^-'s are found to be

$$[\rho_n^-,\ \rho_{n'}^+] = \begin{cases} \sum\limits_{-\frac{n}{2} < \bar{n} < \frac{n'}{2}} \psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}} \psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } -n < n', \\ -\sum\limits_{\frac{n}{2} < \bar{n} < -\frac{n'}{2}} \psi^*_{\bar{n}-\frac{n}{2}-\frac{n'}{2}} \psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}} & \text{for } n < -n', \end{cases} \tag{2.11'}$$

and

$$[\rho_n^-,\ \rho_{-n}^+] = 0. \tag{2.11'}$$

We now show that the commutation relations (2.9)—(2.11′) can be replaced by simpler ones if the states under consideration satisfy, at least approximately, some conditions which will be specified below. These simplified commutation relations tell us that the density field can be regarded as a Bose-field under these restricting conditions.

We first consider the case of the ideal Fermi gas in which there are no interactions between particles. If the gas is not excited too highly, only particles in the neighborhood of the Fermi maximum are raised to higher levels. There exist holes and excited particles only in the neighborhood of the surface of the Fermi sea. Now, in the case of a non-ideal Fermi gas, the inter-particle forces cause virtual transitions of particles. Thus extra holes and exited particles appear. But, if the range of the inter-particle force is not too short and the force itself is not too strong, these virtual holes and excited particles are still present only in the neighborhood of the Fermi maximum. Such are the states to which we will confine ourselves.

If we confine ourselves to states of such type, we can simplify the commutation relations (2.9)—(2.11′) in the following manner.

Let us consider, for instance, the first commutation relations $[\rho_n^+,\ \rho_{n'}^+]$ for which $n > 0$ and $-n < n' < n$. We notice that the expression on the right-hand side is a sum of operators each bringing one particle from the level $\bar{n}+\frac{n}{2}+\frac{n'}{2}$ to the level $\bar{n}-\frac{n}{2}-\frac{n'}{2}$. Because the summation over $\bar{n}$ is extended only between $\frac{n'}{2}$ and $\frac{n}{2}$, the final levels $\bar{n}-\frac{n}{2}-\frac{n'}{2}$ lie in a limited interval between $-\frac{n}{2}$ and $-\frac{n'}{2}$. Now, let $n_{\max}$ denote the value of $|n|$ at the Fermi maximum. Then, if n and $|n'|$ are both sufficiently small compared with $n_{\max}$ (the discussion about how small they should be will be given below), the levels $-\frac{n}{2}$ and $-\frac{n'}{2}$ both lie deep in the bottom of the Fermi sea where there are holes. In such a case the operator $\psi^{*}_{\bar{n}-\frac{n}{2}-\frac{n'}{2}}\psi_{\bar{n}+\frac{n}{2}+\frac{n'}{2}}$ will give a vanishing result because the final level is occupied. Thus, for the states under considration our commutators $[\rho_n^+, \rho_{n'}^+]$ are equivalent to zero.

We next consider $[\rho_n^+, \rho_{-n}^+]$. In this case, the right-hand side is

$$\sum_{-\frac{n}{2}<n\leq\frac{n}{2}} \psi_{\bar{n}}^*\psi_{\bar{n}} = \sum_{-\frac{n}{2}<\bar{n}\leq\frac{n}{2}} N_{\bar{n}}\ , \tag{2.9''}$$

where $N_{\bar{n}}$ is the occupation number of the level $\bar{n}$. Since the level $\bar{n}$, which lies between $-\frac{n}{2}$ and $\frac{n}{2}$, lies deep in the Fermi sea if n is small compared with $n_{\max}$, it is occupied by one particle. Then the sum $\sum N_{\bar{n}}$ is simply equal to the number of levels between $-\frac{n}{2}$ and $\frac{n}{2}$, which is just n. So we find that $[\rho_n^+, \rho_{-n}^+]$ is equivalent to n.

A similar consideration applies to the remaining commutators. We can see that the following commutation relations hold in the sense of equivalence:

$$\begin{aligned} &[\rho_n^+, \rho_{n'}^+]=n\delta_{n,-n'} \\ &[\rho_n^-, \rho_{n'}^-]=-n\delta_{n,-n'} \\ &[\rho_n^+, \rho_{n'}^-]=0. \end{aligned} \tag{2.12}$$

In order that these simpler commutation relations can be used instead of the original ones, it was necessary that $|n|$ and $|n'|$ be sufficiently small compared with $n_{\max}$. We now discass this point more quantitatively.

First we notice that the total number N of the particles is related to $n_{\max}$ by

$$N=2n_{\max}+1, \tag{2.13}$$

which we shall make use of in later considerations.

We now assume that in the states under consideration there are no holes under the level specified by $|n|=\alpha n_{\max}$, α being a positive number less than unity. Then, we can easily see that

$$\frac{3}{2}|n|,\ \frac{3}{2}|n'| < a\, n_{\max} \tag{2.14}$$

is the required condition for the validity of the simplar commutation relations. (The factor 3/2 is required in order that (2.11) be equivalent to the third relation of (2.12)).

So far as the present consideration is concerned, there is no restriction on a. But we shall see later that a must be 3/4 in order that the whole treatment work consistently. The required conditions will be found to be the following:

(I) In the region $|n| < \frac{3}{4} n_{\max}$, there should be no holes.

(II) In the region $|n| > \frac{5}{4} n_{\max}$, there should be no excited particles.

(III) The absolute values of n in ρ_n^+ and ρ_n^- should not exceed $\frac{1}{2} n_{\max}$:

$$|n| < \frac{1}{2} n_{\max}. \tag{2.15}$$

Our method works when and only when these conditions are satisfied. The conditions (I) and (II) restrict the states; therefore, it is always necessary in applying our method to verify whether the states obtained as an answer do really satisfy these conditions. The condition (III) requires that no sound waves having shorter wave length than $2L/n_{\max}$ play a role in the problem. This condition will be satisfied if the range of inter-particle force is sufficiently long. The reason for the the necessity of the special choice of $a=3/4$ will be given later.

§ 3. Equation of motion and Hamiltonian for the density field.

We first consider the case of non-interacting particles. In this case the change with time of ρ_n^+ and ρ_n^- can be obtained easily because we know the change with time of ψ_n and $\psi_n{}^*$. In the general case we them find a very complicated time dependence of the ρ's. The motions of ρ_n^+ and ρ_n^- are by no means simple harmonic because each term of $\sum\psi^*_{\bar{n}-\frac{n}{2}}\,\psi_{\bar{n}+\frac{n}{2}}$ has a frequency which depends not only on n but also on $\bar{n}$. This fact means that the density does by no means behave like waves.

However, we may here make use of our conditions imposed on the states. According to them holes and excited particles are present in a narrow interval only, from $(3/4)n_{\max}$ to $(5/4)n_{\max}$, in the neighborhood of $n_{\max}$. This results in non-vanishing matrix elements in $\sum\psi^*_{\bar{n}-\frac{n}{2}}\,\psi_{\bar{n}+\frac{n}{2}}$ being contributed solely by terms with $\bar{n}\pm\frac{n}{2}$ nearly equal to $n_{\max}$ or $-n_{\max}$ according as we are dealing with ρ_n^+ or ρ_n^-. We may then approximate the frequencies of ψ^* and ψ by their expansion in the neighborhood of $\bar{n}\pm\frac{n}{2}=n_{\max}$ or $\bar{n}\pm\frac{n}{2}=-n_{\max}$ respectively. The frequency of $\psi_{\bar{n}+\frac{n}{2}}$ is, for instance,

$$\frac{1}{2\pi\hbar}\frac{1}{2m}\left(\frac{2\pi\hbar}{L}\right)^2\left(\bar{n}+\frac{n}{2}\right)^2,$$

m being the mass of the particles. If we here expand $\left(\bar{n}+\frac{n}{2}\right)^2$ in the neighborhood of $\bar{n}+\frac{n}{2}=n_{\max}$, we get

$$\left(\bar{n}+\frac{n}{2}\right)^2=\left\{n_{\max}+\left(\bar{n}+\frac{n}{2}-n_{\max}\right)\right\}^2$$
$$=n_{\max}^2+2n_{\max}\left(\bar{n}+\frac{n}{2}-n_{\max}\right)+\cdots.$$

Neglecting the small term $\left(\bar{n}+\frac{n}{2}-n_{\max}\right)^2$, the time dependence of $\psi_{\bar{n}+\frac{n}{2}}$ becomes

$$\psi_{\bar{n}+\frac{n}{2}}\approx\exp i\left[+\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}^2}{2m}-\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left(\bar{n}+\frac{n}{2}\right)\right]t.$$

In the same way we have

$$\psi_{\bar{n}-\frac{n}{2}}\approx\exp i\left[-\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}^2}{2m}+\frac{1}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left(\bar{n}-\frac{n}{2}\right)\right]t.$$

Then, combining these two, we get a time dependence for ρ_n^+ of the form

$$\rho_n^+\approx\exp\left[-\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\right]t. \tag{3.1}$$

By the same consideration using the expansion in the neighborhood of $\bar{n}\pm\frac{n}{2}=-n_{\max}$, we get

$$\rho_n^-\approx\exp\left[+\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\right]t. \tag{3.2}$$

From (3.1) and (3.2) we see that ρ^+ and ρ^- satisfy the equations of motion

$$\begin{cases}\dot{\rho}_n^+=-\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\rho_n^+\\ \dot{\rho}_n^-=+\frac{i}{\hbar}\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}n\rho_n^-.\end{cases} \tag{3.3}$$

We notice here that the commutation relations (2.12) lead to the fact that the canonically conjugate momentum for

$$\rho_n=\rho_n^++\rho_n^- \tag{3.4}$$

is given by

$$\pi_n = \frac{i}{2}\frac{1}{n}\left(\rho_{-n}^{+} - \rho_{-n}^{-}\right). \tag{3.5}$$

As can be easily verified, we see that

$$[\rho_n,\ \pi_{n'}] = i\delta_{n,n'}. \tag{3.6}$$

By the equation of motion we then find

$$\pi_n = \frac{\hbar}{2}\left(\frac{L}{2\pi\hbar}\right)^2 \frac{m}{n_{\max}} \frac{1}{n^2} \dot{\rho}_{-n}. \tag{3.5$'$}$$

The last relation gives a physical meaning to our canonical momenta: As usual, momenta are proportional to the time derivatives of the coordinates. Though the equations of motion of the form of (3.3) hold only for the case of non-interacting particles, we shall see later that (3.5′) holds more generally.

We shall now set up the Hamiltonian for the ρ field. The Hamiltonian is determined by the requirement that

$$\frac{\hbar}{i}\dot{\rho}_n^{\pm} = [\mathfrak{H},\ \rho_n^{\pm}] \tag{3.7}$$

yield the equations of motion (3.3) wtih the help of the commutation relations (2.12). We then find that the Hamiltonian has a very simple form:

$$\mathfrak{H} = \left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m} \sum_{n>0} \left(\rho_{-n}^{+}\rho_n^{+} + \rho_n^{-}\rho_{-n}^{-}\right). \tag{3.8}$$

The order of non-commuting factors is here specified by the condition that $\mathfrak{H}$ has no zero point value.

Though we have set up the Hamiltonian in this way, it is by no means self-evident that this Hamiltonian gives in fact the energy of the system (or eventually energy plus some additive constant). That this is actually the case will be proved in the next section.

So far we have neglected the mutual interaction between particles. It is quite simple to introduce the interaction into the theory, when the interaction force is of the ordinary type, neither of exchange type nor velocity dependent. In this case the interaction energy has the form

$$H_{\text{int}} = \frac{1}{2}\iint \rho(x)\rho(x')J(|x-x'|)\,dx\,dx' - \frac{1}{2}\int \rho(x)J(0)\,dx, \tag{3.9}$$

$J(|x-x'|)$ being the potential of the inter-particle force. The term $\frac{1}{2}\int \rho(x)J(0)\,dx$ is subtracted in order to remove the interaction of a particle with itself. In terms of the Fourier transform we find

$$H_{\text{int}} = \frac{1}{2}\sum_n{}' \rho_n\rho_{-n}J_n + \frac{1}{2}\rho_0^2 J_0 - \frac{1}{2}\rho_0 J(0), \tag{3.10}$$

where J_n is the matrix element of the potential defined by

$$J_n=\frac{1}{L}\int J(x)\exp\left(\frac{2\pi i}{L}nx\right)dx=J_{-n}. \tag{3.11}$$

The prime on the $\sum'$ symbol means that the term with $n=0$ should be omitted. ρ_0 is the quantity defined by

$$\rho_0=\sum_{\bar{n}}\psi_{\bar{n}}\psi_{\bar{n}}=N, \tag{3.12}$$

i.e. the total number of particles.

Since H_{int} commutes with each ρ_n, which is the sum $\rho_n^- + \rho_n^-$, the equation of motion for ρ_n is not affected by the interaction, while the equations of motion for ρ_n^+ and ρ_n^- separately are affected by the interaction. This fact means that the canonically conjugate momenta π_n are still given by (3.5′) irrespective of the presence or absence of the interaction.

In terms of ρ^+ and ρ^- the interaction Hamiltonian can be expressed as

$$\begin{aligned}H_{\text{int}}=&\sum_{n>0}J_n(\rho_{-n}^+\rho_n^+ + \rho_n^-\rho_{-n}^- + \rho_n^+\rho_{-n}^- + \rho_n^-\rho_{-n}^+)\\&+\sum_{n>0}nJ_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0).\end{aligned} \tag{3.13}$$

The term $\sum\limits_{n>0}nJ_n$ appeared when we performed the rearrangement of factors in $\rho_n^+\rho_{-n}^+$ and $\rho_{-n}^-\rho_n^-$ into the correct order.

It is the essential point of our method that the energy, the "kinetic part" $\mathfrak{H}$ as well as the "potential part" H_{int}, is bilinear in the $\rho^\pm$'s. This fact is in marked contrast to the fact that, in terms of ψ and ψ^*, only the kinetic part is bilinear; the potential part contains four ψ's. This latter fact makes it necessary to solve a complicated non-linear problem if one wishes to deal with a many-particle problem with inter-particle force. Now that we have found the Hamiltonian to be bilinear in the ρ's, irrespective of the presence or absence of the inter-particle force, we have no such difficulty if we deal with the problem in terms of the ρ field not in terms of the ψ field. The problem is simply to perform the principal-axes transformation of the bilinear form, i. e. to find the normal coordinates for the ρ field.

Before we enter into this problem, we shall show that the Hamiltonian $\mathfrak{H}$ is in fact equal to the kinetic energy of the system minus some constant which depends only on the total number of particles.

§ 4. Energy and momentum of the system.

In this section we give the proof that the Hamiltonian $\mathfrak{H}$ really gives the kinetic energy of the system. This can be done by a straightforward calculation in the following manner.

For this purpose we first examine $\sum\rho_{-n}^+\rho_n^+$. We examine the diagonal and the non-diagonal parts of $\sum \rho_{-n}^{+} \rho_{n}^{+}$ separately, where the matrix is supposed to be referred to in the representation in which the occupation numbers of the particles are diagonal.

We have

$$\begin{aligned}\sum_{\frac{n_{\max}}{2}>n>0} \rho_{-n}^{+}\rho_{n}^{+} &= \sum_{\frac{n_{\max}}{2}>n>0} \sum_{\bar{n}>0} \sum_{\bar{n}'>0} \psi^{*}_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^{*}_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}} \\ &= \sum_{\frac{n_{\max}}{2}>n>0} \Big(\sum_{\bar{n}>0} \sum_{\bar{n}'>0}\Big)' \psi^{*}_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^{*}_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}} \\ &\quad + \sum_{\frac{n_{\max}}{2}>n>0} \sum_{\bar{n}>0} (1-N_{\bar{n}-\frac{n}{2}}) N_{\bar{n}+\frac{n}{2}}. \end{aligned} \tag{4.1}$$

The prime on $(\sum \sum)'$ means that terms with $\bar{n}=\bar{n}'$ are omitted. By this omission the first sum on the right-hand side of (4.1) gives the non-diagonal part and the second sum the diagonal part of $\sum \rho_{-n}^{+} \rho_{n}^{+}$. The summation over n is to be extended only up to $n_{\max}/2$ because of our condition (2.15) (III).

We first observe the diagonal part

$$D^{+} = \sum_{\frac{n_{\max}}{2}>n>0} \sum_{\bar{n}>0} (1-N_{\bar{n}-\frac{n}{2}}) N_{\bar{n}+\frac{n}{2}}. \tag{4.2}$$

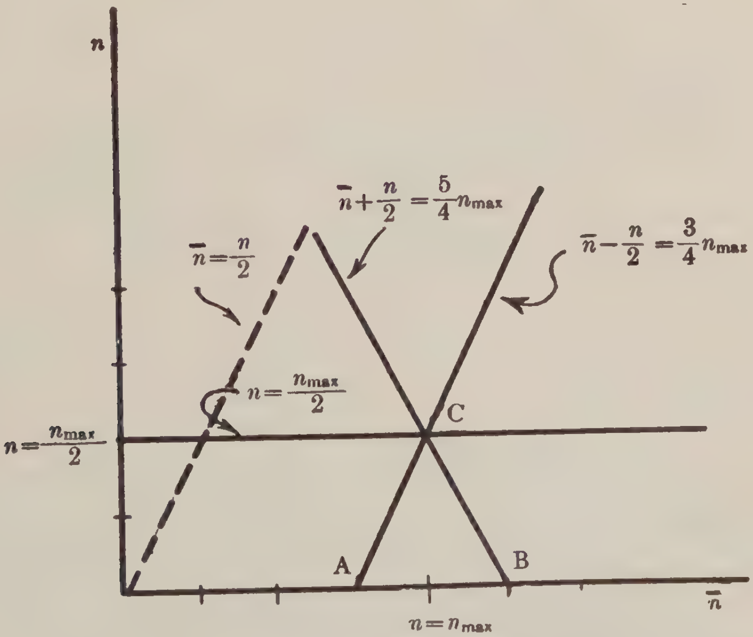

Fig. 1.

We make use of the fact that, because of the first two conditions of (2.15), the summand $(1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}}$ has a non-vanishing value only at points $(n, \bar{n})$ lying inside the triangle ABC (see Fig. 1) enclosed by the three straight lines

$$\begin{cases} \bar{n}+\dfrac{n}{2}=\dfrac{5}{4}n_{\max} \\ \bar{n}-\dfrac{n}{2}=\dfrac{3}{4}n_{\max} \\ n=0\ . \end{cases}$$

Since the summand vanishes outside this triangle, two of the boundaries of the domain of summation

$$n=\frac{n_{\max}}{2} \qquad \text{and} \qquad \bar{n}=0 \tag{4.3}$$

can be changed at our will so long as the changed boundaries lie outside the triangle. So we may use the boundaries

$$n=\infty \qquad \text{and} \qquad \bar{n}=\frac{n}{2} \tag{4.3'}$$

instead of the original ones. Then we have

$$\begin{aligned} D^{+}&=\sum_{\infty>n>0}\ \sum_{\bar{n}\geqq\frac{n}{2}}\ (1-N_{\bar{n}-\frac{n}{2}})N_{\bar{n}+\frac{n}{2}} \\ &=\sum_{n>0}\ \sum_{\bar{n}'\geqq n}\ (1-N_{\bar{n}'-n})N_{\bar{n}'}. \end{aligned} \tag{4.4}$$

We now notice that

$$\sum_{n>0}\ \sum_{\bar{n}'\geqq n}\ N_{\bar{n}'}=\sum_{\bar{n}'\geqq 0}\bar{n}'\,N_{\bar{n}'} \tag{4.5}$$

and

$$\begin{aligned} \sum_{n>0}\ \sum_{\bar{n}'\geqq n}\ N_{\bar{n}'-n}\ N_{\bar{n}'}&=\frac{1}{2}\Big(\sum_{\bar{n}\geqq 0}N_{\bar{n}}\Big)^{2}-\frac{1}{2}\sum_{\bar{n}\geqq 0}N_{\bar{n}} \\ &=\frac{1}{2}n_{\max}\ (n_{\max}+1). \end{aligned} \tag{4.5'}$$

We then get from (4.4)

$$D^{+}=\sum_{n\geqq 0}\ nN_{n}-\frac{1}{2}n_{\max}\ (n_{\max}+1). \tag{4.6}$$

In the same way we get the corresponding result for $\sum\ \rho_{\bar{n}}\rho_{-\bar{n}}$

$$D^{-}=\sum_{n\leqq 0}\ |n|\,N_{n}-\frac{1}{2}n_{\max}\ (n_{\max}+1). \tag{4.7}$$

We now go over to the non-diagonal part:

$$C^{+}=\sum_{\frac{n_{\max}}{2}>n>0}\ \Big(\sum_{\bar{n}>0}\ \sum_{\bar{n}'>0}\Big)'\ \psi^{*}_{\bar{n}+\frac{n}{2}}\ \psi_{\bar{n}-\frac{n}{2}}\ \psi^{*}_{\bar{n}'-\frac{n}{2}}\ \psi_{\bar{n}'+\frac{n}{2}} \tag{4.8}$$

and show that C^{+} vanishes.

By writing $-n$ instead of n, $\bar{n}'$ instead of $\bar{n}$ and $\bar{n}$ instead of $\bar{n}'$, we see that C^+ can be written also in the following form:

$$C^+ = \sum_{-\frac{n_{\max}}{2}<n<0} \Big(\sum_{\bar{n}>0} \sum_{\bar{n}'>0} \Big)' \psi^*_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^*_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}}, \tag{4.8}$$

where we have used the fact that $\psi^*_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}}$ commutes with $\psi^*_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}}$ since $\bar{n} \neq \bar{n}'$. From (4.8) and (4.8′) we get

$$C^+ = \frac{1}{2} \sum'_{-\frac{n_{\max}}{2}<n<\frac{n_{\max}}{2}} \Big(\sum_{\bar{n}>0} \sum_{\bar{n}'>0} \Big)' \psi^*_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^*_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}}. \tag{4.8''}$$

Now a considerasion similar to the above one shows that, because of the condition (2.15), we can change the boundaries of summation

$$n = \pm \frac{n_{\max}}{2}, \quad \bar{n}=0 \quad \text{and} \quad \bar{n}'=0 \tag{4.9}$$

into new ones

$$n = \pm\infty, \qquad \bar{n} = -\frac{n}{2}, \qquad \bar{n}' = \frac{n}{2}. \tag{4.9'}$$

We then get

$$C^+ = \frac{1}{2} \sum'_{-\infty<n<\infty} \Big(\sum_{\bar{n}\geqq-\frac{n}{2}} \sum_{\bar{n}'\geqq\frac{n}{2}} \Big)' \psi^*_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^*_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}}; \tag{4.10}$$

but this C^+ vanishes as is shown in the following manner.

By rearranging factors in (4.10) we get

$$C^+ = -\frac{1}{2} \sum'_{-\infty<n<\infty} \Big(\sum_{\bar{n}\geqq-\frac{n}{2}} \sum_{\bar{n}'\geqq\frac{n}{2}} \Big)' \psi^*_{\bar{n}'-\frac{n}{2}} \psi_{\bar{n}-\frac{n}{2}} \psi^*_{\bar{n}+\frac{n}{2}} \psi_{\bar{n}'+\frac{n}{2}}. \tag{4.10'}$$

If we put here

$$\begin{cases} l = \bar{n}' - \bar{n} \\ \bar{l}' = \dfrac{1}{2}(\bar{n}' + \bar{n} + n) \\ \bar{l} = \dfrac{1}{2}(\bar{n}' + \bar{n} - n), \end{cases}$$

we can write (4.10′) in the form:

$$C^+ = -\frac{1}{2} \sum'_{-\infty<l<\infty} \Big(\sum_{\bar{l}\geqq-\frac{l}{2}} \sum_{\bar{l}'\geqq\frac{l}{2}} \Big)' \psi^*_{\bar{l}+\frac{l}{2}} \psi_{\bar{l}-\frac{l}{2}} \psi^*_{\bar{l}'-\frac{l}{2}} \psi_{\bar{l}'+\frac{l}{2}}. \tag{4.11}$$

That the domains of summation over l, $\bar{l}$ and $\bar{l}'$ are really as specified by the inequalities under the summation symbols can be easily shown. One has only to notice the identities

$$\begin{cases} \bar{n}+\dfrac{n}{2}=\bar{l}'-\dfrac{l}{2} \\ \bar{n}'-\dfrac{n}{2}=\bar{l}+\dfrac{l}{2}. \end{cases}$$

Comparing (4.11) with (4.10) we find $C^{+}=-C^{+}$, which means that

$$C^{+}=0. \tag{4.12}$$

A similar consideration applies to $\sum \rho_{n}^{-}\rho_{-n}^{-}$. We find that the non-diagonal part of $\sum \rho_{n}^{-}\rho_{-n}^{-}$ vanishes:

$$C^{-}=0. \tag{4.13}$$

This rather lengthy proof that $C^{+}=C^{-}=0$ can be summarized in the following way. Each term in C^{+} or C^{-} corresponds to a simultaneous transition of two particles. If there is a term corresponding to a transition $a\rightarrow b$, $c\rightarrow d$, there is always a term corresponding to the transition $a\rightarrow d$, $c\rightarrow b$. Because of the Fermi statistics, the latter term has a sign which is opposite to the former so that they cancel each other when added together.

Substituting (4.6), (4.7), (4.12) and (4.13) into $\mathfrak{H}$, we obtain

$$\begin{aligned} \mathfrak{H}&=\left(\frac{2\pi\hbar}{L}\right)^{2}\frac{n_{\max}}{m}\{\sum_{n}|n|\,N_{n}-n_{\max}(n_{\max}+1)\} \\ &=\left(\frac{2\pi\hbar}{L}\right)^{2}\frac{n_{\max}}{m}\{\sum_{|n|\leqq n_{\max}}|n|\,(N_{n}-1)+\sum_{|n|>n_{\max}}|n|\,N_{n}\}. \end{aligned} \tag{4.14}$$

On the other hand the kinetic energy of the system is evidently

$$H_{\text{kin}}=\left(\frac{2\pi\hbar}{L}\right)^{2}\frac{1}{2m}\sum_{n}n^{2}N_{n}. \tag{4.15}$$

In the state of perfect degeneracy, i. e., in the state where all levels between $-n_{\max}$ and $n_{\max}$ are occupied and all other levels are empty, this energy becomes

$$H_{0}=\left(\frac{2\pi\hbar}{L}\right)^{2}\frac{1}{2m}\sum_{|n|\leqq n_{\max}}n^{2}. \tag{4.16}$$

We shall now show that, in the same approximation, which is allowed by our condition (2.15), our Hamiltonian $\mathfrak{H}$ is equal to the energy (4.15) minus the constant energy (4.16) : $\mathfrak{H}=H_{\text{kin}}-H_{0}$.

We calculate $H_{\text{kin}}-H_{0}$ in the following way: From (4.15) and (4.16) we get

$$H_{\text{kin}}-H_{0}=\left(\frac{2\pi\hbar}{L}\right)^{2}\frac{1}{2m}\{\sum_{|n|\leqq n_{\max}}n^{2}(N_{n}-1)+\sum_{|n|>n_{\max}}n^{2}N_{n}\}. \tag{4.17}$$

Making use of the assumption that holes and excited particles are present only in the neighborhood of $\pm n_{\max}$, we see that $N_{n}-1$ and N_{n} differ from zero only when n lies in the neighborhood of $\pm n_{\max}$. This allows us to use instead of n^2 its expansion:

$$\begin{aligned} n^2 &= \{\pm n_{\max} + (n \mp n_{\max})\}^2 \\ &\approx n_{\max}^2 \pm 2n_{\max}(n \mp n_{\max}) \\ &= -n_{\max}^2 + 2n_{\max}|n|. \end{aligned}$$

Substituting this in (4.17), we find

$$H_{\mathrm{kin}} - H_0 = -\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}^2}{m}\{N-(2n_{\max}+1)\} + \mathfrak{H}. \tag{4.17'}$$

Then the use of (2.13) gives immediately the required result:

$$H_{\mathrm{kin}} - H_0 = \mathfrak{H}. \tag{4.18}$$

This result means that our Hamiltonian $\mathfrak{H}$ is really the kinetic energy of the assembly minus a constant which depends only on the total number of particles. This constant is the energy of the assembly in the case of perfect degeneracy and thus $\mathfrak{H}$ can be interpreted as the deviation of the kinetic energy from this standard value. We express this fact by simply calling $\mathfrak{H}$ " the excitation kinetic energy ".

The total energy of the system is evidently

$$H = \mathfrak{H} + H_{\mathrm{int}} + H_0 \tag{4.19}$$

when there are inter-particle forces.

The relations (4.6) and (4.7) can be used to express the momentum of the assembly in terms of ρ. Let the momentum be denoted by G. Then

$$G = \left(\frac{2\pi\hbar}{L}\right) \sum_{n>0} (\rho_{-n}^{+} \rho_{n}^{+} - \rho_{n}^{-} \rho_{-n}^{-}). \tag{4.20}$$

The expression for G applies irrespective of the presence or absence of the inter-particle force.

One sees that for the proof of the relation (4.18) the possibility of change of domains of summation, the replacement of (4.3) by (4.3′) and the replacement of (4.9) by (4.9′), were essential. These are possible only when the three conditions in (2.15) are simultaneously satisfied. This is the reason why we had to choose the special value of $a=3/4$. It is easy to see that another choice of a would make it impossible to change the summation domains so that we would not get the relation (4.18).

§ 5. Solution of the eigen-value problem.

We now go over to the solution of the eigen-value problem of our Hamiltonian. As mentioned before, this can be done by performing the principal-axes transformation of the bilinear form representing the Hamiltonian.

Adding (3.8), (3.13) and H_0, the total Hamiltonian is obtained:

$$
\begin{aligned}
H&=\mathfrak{H}+H_{\mathrm{int}}+H_0\\
&=\sum_{n>0}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+J_n\right\}(\rho_{-n}^{+}\rho_n^{+}+\rho_n^{-}\rho_{-n}^{-})\\
&\quad+\sum_{n>0}J_n(\rho_n^{+}\rho_{-n}^{-}+\rho_n^{-}\rho_{-n}^{+})\\
&\quad\quad+\sum_{n>0}nJ_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0.
\end{aligned}
\tag{5.1}
$$

We introduce real coordinates Q_n and real momenta P_n by means of

$$
\left\{
\begin{aligned}
\rho_n^{+}&=\sqrt{\frac{n}{2}}(Q_n+iP_n)\\
\rho_{-n}^{+}&=\sqrt{\frac{n}{2}}(Q_n-iP_n)\\
\rho_n^{-}&=\sqrt{\frac{n}{2}}(Q_{-n}-iP_{-n})\\
\rho_{-n}^{-}&=\sqrt{\frac{n}{2}}(Q_{-n}+iP_{-n}),
\end{aligned}
\right.
$$

where the suffix n is considered positive. Then in terms of P's a ' Q's the Hamiltonian is expressed as

$$
\begin{aligned}
H=\sum_n|n|&\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+J_n\right\}\left(\frac{1}{2}P_n^2+\frac{1}{2}Q_n^2-\frac{1}{2}\right)\\
&+\frac{1}{2}\sum_n|n|J_n(Q_nQ_{-n}-P_nP_{-n})\\
&\quad+\frac{1}{2}\sum_n|n|J_n+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0.
\end{aligned}
\tag{5.3}
$$

In this expression the suffix n takes both positive and negative values and the summation $\sum_n$ is extended over positive as well as negative valus of n.

The commutation relations for the P's and Q's are

$$
\left\{
\begin{aligned}
&[Q_n,P_{n'}]=i\,\delta_{n,n'}\\
&[Q_n,Q_{n'}]=[P_n,P_{n'}]=0.
\end{aligned}
\right.
\tag{5.4}
$$

The transformation of H into principal form can be performed by defining new variables q_n and p_n by means of the canonical transformation

$$
\begin{aligned}
P_n&=\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}+\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}P_n+\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}-\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}p_{-n}\\
Q_n&=\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}+\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}q_n-\frac{1}{2}\left\{\left(\frac{T_n+2U_n}{T_n}\right)^{1/4}-\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\right\}q_{-n}
\end{aligned}
\tag{5.5}
$$

with the abbreviations

$$\begin{cases} T_n=\left(\dfrac{2\pi\hbar}{L}\right)^2\dfrac{n_{\max}}{m}\,|n|, \\ U_n=|n|\,J_n\,. \end{cases} \tag{5.6}$$

It can be easily seen that the p_n and q_n satisfy

$$\begin{cases} [q_n, p_{n'}]=i\,\delta_{n,n'}\,, \\ [q_n, q_{n'}]=[p_n, p_{n'}]=0, \end{cases} \tag{5.7}$$

and that H is transformed into

$$H=\sum_n\sqrt{T_n}\left\{\sqrt{T_n+2U_n}\left(\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}\right)+\frac{1}{2}\left(\sqrt{T_n+2U_n}-\sqrt{T_n}\right)\right\}$$
$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,. \tag{5.8}$$

The momentum G, expressed in terms of q's and p's, is

$$G=\left(\frac{2\pi\hbar}{L}\right)\sum_n n\left(\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}\right). \tag{5.9}$$

The expressions (5.8) and (5.9) show that our system is in fact equivalent to an assembly of uncoupled harmonic oscillators, or, better, to an assembly of uncoupled sound quanta. Each sound quantum has momentum and energy given by

$$\text{momentum}=\left(\frac{2\pi\hbar}{L}\right)n \qquad n=0,\ \pm 1,\ \pm 2,\cdots \tag{5.10}$$

and

$$\begin{aligned}\text{energy}&=\sqrt{T_n(T_n+2U_n)}\\ &=\left[\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}|n|,\end{aligned} \tag{5.10$'$}$$

and the number of sound quanta with the momentum $\left(\dfrac{2\pi\hbar}{L}\right)n$ is represented by the operator

$$M_n=\frac{1}{2}p_n^2+\frac{1}{2}q_n^2-\frac{1}{2}. \tag{5.11}$$

In terms of M_n the energy and the momentum are

$$H=\sum_n\sqrt{T_n}\left\{\sqrt{T_n+2U_n}\;M_n+\frac{1}{2}\left(\sqrt{T_n+2U_n}-\sqrt{T_n}\right)\right\} \tag{5.8$'$}$$
$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,,$$

and

$$G=\left(\frac{2\pi\hbar}{L}\right)\sum_{n} n\, M_n \tag{5.9'}$$

respectively.

In the wave language the relations (5.10) and (5.11′) state that the sound wave with the wave length

$$\lambda_n=\frac{L}{|n|} \tag{5.12}$$

has the frequency

$$\nu_n=\frac{1}{2\pi}\left[\left(\frac{2\pi}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}|n|\,. \tag{5.12'}$$

The phase velocity of the sound wave is

$$v_n'=\left[\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}\left\{\left(\frac{2\pi\hbar}{L}\right)^2\frac{n_{\max}}{m}+2J_n\right\}\right]^{1/2}. \tag{5.13}$$

The phase velocity given by (5.13) is dependent on n and hence on λ_n when J_n does not vanish. This means that, when the inter-particle force is present, our Fermi gas is dispersive. (5.13) shows further that the phase velocity of the sound waves increases or decreases from its value in the ideal gas according as the inter-particle force is repulsive or attractive.

The formula (5.8′) enables us immediately to calculate the energy in various stationaly states. In the lowest state, for instance, where there are no sound quanta, we get

$$E=\frac{1}{2}\sum_{n}\sqrt{T_n}\left(\sqrt{T_n+2U_n}-\sqrt{T_n}\right)$$

$$+\frac{1}{2}N^2J_0-\frac{1}{2}NJ(0)+H_0\,. \tag{5.14}$$

In the case of an attractive inter-particle force it may happen that E, considered as a function of L, has a minimum for some value of L. Then this will mean that the assembly is capable of forming a stable aggregate.

In the case where J_n is very small, we can expand $\sqrt{T_n+2U_n}$ and thus express E as a power series in J_n. We shall then find that the term linear in J_n together with the term H_0 gives the energy in the usual Tomas-Fermi ap-approximation including the effect of exchange. The energy quadratic in J_n corresponds to the energy obtained by Euler as the second order term of a perturbation calculation.

§ 6. Correlation between particles. Transitions caused by an external perturbing force. Criterion for applicability of the method.

In this section we shall mention some general results of physical interest which can be obtained directly by our theory. We shall give also a criterion for the applicability of our method.

(I) Correlation of position of particles.

As is known there is no correlation between particles in an ideal gas besides that due to the exclusion principle. The interaction between particles causes an extra correlation, which can be calculated very simply.

Mathematically the correlation of position can be exppessed by the quantity

$$C(\xi)=\frac{1}{N^2}\left\langle\int\rho(x)\rho(x+\xi)\,dx\right\rangle_{Av}, \tag{6.1}$$

$C(\xi)d\xi$ giving the probability of finding another particle at a distance between ξ and $\xi+d\xi$ from a particle. This correlation function can be calculated in the following manner.

Expressing (6.1) in terms of the Fourier transform of $\rho(x)$, we have

$$C(\xi)=\frac{1}{L}+\frac{1}{2N^2L}\sum_{n>0}\langle\rho_n\rho_{-n}\rangle_{Av}\cos\left(\frac{2\pi n}{L}\xi\right), \tag{6.2}$$

so that our task is to find $\langle\rho_n\rho_{-n}\rangle_{Av}$. in terms of the q's and p's defined by (5.2) and (5.5) the operator $\rho_n\,\rho_{-n}$ is

$$\rho_n\,\rho_{-n}=2n\sqrt{\frac{T_n}{T_n+2U_n}}\left\{\frac{1}{2}(p_n^2+q_n^2)+\frac{1}{2}(p_{-n}^2+q_{-n}^2)+q_n\,q_{-n}-p_n\,p_{-n}\right\}. \tag{6.3}$$

It then follows that

$$\langle\rho_n\rho_{-n}\rangle_{Av}=2n\sqrt{\frac{T_n}{T_n+2U_n}}(M_n+M_{-n}+1). \tag{6.4}$$

which gives

$$C(\xi)=\frac{1}{L}+\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)\sqrt{\frac{T_n}{T_n+2U_n}}\,|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.5}$$

The correlation function $C(\xi)$ consists of two parts. The one is the part expressing the correlation which already exists in the absence of the inter-particle force. This part is due solely to the exclusion principle. This part of the correlation can be obtained by putting $U_n=0$ in (6.5). Denoting this part of $C(\xi)$ by $C_0(\xi)$, we have

$$C_0(\xi)=\frac{1}{L}+\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.6}$$

The second part of $C(\xi)$ is that part of the correlation which is purely due to inter-particle forces. Denoting this second part by $C_1(\xi)$, we get

$$C_1(\xi)=\frac{1}{N^2L}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)|n|\cos\left(\frac{2\pi n}{L}\xi\right). \tag{6.7}$$

(6.7) can also be written as

$$C_1(\xi)=\frac{\partial}{\partial\xi}\left\{\frac{1}{2\pi N^2}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)\sin\left(\frac{2\pi|n|}{L}\xi\right)\right\}. \tag{6.7'}$$

Then the quantity

$$D_1(\xi)=\frac{1}{2\pi N^2}\sum_n\left(M_n+\frac{1}{2}\right)\left(\sqrt{\frac{T_n}{T_n+2U_n}}-1\right)\sin\left(\frac{2\pi|n|}{L}\xi\right) \tag{6.7''}$$

can be interpreted as the change of probability due to the inter-particle force of finding another particle within the distance ξ from a particle.

(II) Matrix element of an external perturbing force.

Suppose an external perturbing force is impressed upon the system. Then the perturbation will cause transitions of the system. As is well known, the transition probabilities depend essentially on the matrix elements of the perturbing energy. Now, it is very simple to calculate these matrix elements.

Let the potential of the perturbing force be $V(x)$. Then the perturbation energy is

$$\begin{aligned}H'&=\int\rho(x)V(x)\,dx\\&=\sum_{n>0}\{V_n(\rho_n^++\rho_n^-)+V_{-n}(\rho_{-n}^++\rho_{-n}^-)\},\end{aligned} \tag{6.8}$$

where V_n and V_{-n} are the matrix elements of $V(x)$ defined by

$$\left\{\begin{aligned}&V_n=\frac{1}{L}\int V(x)\,\exp\left(\frac{2\pi i}{L}nx\right)dx\\&V_{-n}=V_n^*\,.\end{aligned}\right. \tag{6.9}$$

In terms of the q's and p's it is found that

$$\begin{aligned}H'=\sum_{n>0}\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}\Bigg[&\sqrt{\frac{n}{2}}\,V_n\{(q_n+ip_n)+(q_{-n}-ip_{-n})\}\\&+\sqrt{\frac{n}{2}}\,V_{-n}\{(q_{-n}+ip_{-n})+(q_n-ip_n)\}\Bigg].\end{aligned} \tag{6.10}$$

Now, the matrix elements of $q_{\pm n}+ip_{\pm n}$ as well as $q_{\pm n}-ip_{\pm n}$ are all known. They are the matrix elements for harmonic oscillators. Our result (6.10) shows that the matrix elements of H' are given simply by multiplying these universal matrix elements by the factor $(T_n/T_n+2U_n)^{1/4}\sqrt{n/2}\,V_{\pm n}$. When the inter-particle force is absent, the factors are simply $\sqrt{n/2}\,V_{\pm n}$. Our result shows that the effect of the inter-particle force is simply to replace $\sqrt{n/2}\,V_{\pm n}$ by $(T_n+2U_n)^{1/4}\times\sqrt{n/2}\,V_{\pm n}$. The effect of the inter-particle force can thus be dealt with as an apparent change of the perturbing potential by the factor $(T_n/T_n+2U_n)^{1/4}$, the effective potential being expressed as

$$V^{\mathrm{eff}}_{\pm n} = \left(\frac{T_n}{T_n+2U_n}\right)^{1/4} V_{\pm n}. \tag{6.11}$$

The factor $\left(\frac{T_n}{T_n+2U_n}\right)^{1/4}$ is larger or smaller than unity according as the inter-particle force is attractive or repulsive. This means that in the case of an attractive inter-particle force the effect of an external perturbing force becomes larger, and in the case of a repulsive one the perturbing force becomes less effective as compared with the case of an ideal gas.

(III) Criterion for the applicability of the method.

Because our theory is based on the fundamental conditions of (2.15), it is necessary to see under what circumstances we can be sure that these conditions are satisfied, at least approximately.

It is first clear that the range of the inter-particle force should not be too short in order that the third condition of (2.15) be satisfied. This condition requires that sound waves with wave lengths shorter than $2L/n_{\max}$ play no role. Let us fist consider the lowest state. Then this condition requires that in the sum $\sum_n \sqrt{T_n}\,(\sqrt{T_n+2U_n}-\sqrt{T_n})$ of (5.14) no contribution should arise from terms with $n > n_{\max}/2$. This requires that J_n is negligibly small for $n > n_{\max}/2$. Thus, roughly speaking, the range of the force should be longer than $2L/n_{\max}$ or larger than four times the mean distance of particles.

In the excited states we have to impose a further condition: There should not be any sound quanta with wave length shorter than $2L/n_{\max}$.

In order to see whether or not the first two conditions of (2.15) are satisfied, it is necessary to examine the eigen function which is obtained and see how the holes and excited particles are distributed among the levels. If this eigen function is such that the probability distribution of holes and excited particles is essentially limited between the levels $(3/4)n_{\max}$ and $(5/4)n_{\max}$, the solution obtained will be a good approximation. There is a rough but simple criterion to see under what circumstances such will be the case.

It is to calculate the mean excitation kinetic energy per particle. If this energy is too large, we must expect that there will be a large number of highly excited particles and many holes lying deep in the Fermi sea. If this energy is small, we are sure that the excited particles and holes are present only in the neighborhood of the Fermi maximum.

The excitation kinetic energy per particle is given by

$$\frac{1}{N}\langle \mathfrak{H}\rangle_{Av} = \frac{1}{2n_{\max}+1}\left(\frac{2\pi\hbar}{L}\right)^2 \frac{n_{\max}}{m} \sum_{n>0} \langle \rho^{+}_{-n}\rho^{+}_{n} + \rho^{-}_{n}\rho^{-}_{-n}\rangle_{Av}, \tag{6.12}$$

where we have

$$\rho_{-n}^{+}\rho_{n}^{+}+\rho_{n}^{-}\rho_{-n}^{-}=\frac{n}{2}\Big[\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}\Big)\Big\{\frac{1}{2}(p_n^2+q_n^2)+\frac{1}{2}(p_{-n}^2+q_{-n}^2)\Big\}$$

$$+\Big(\sqrt{\frac{T_n+2U_n}{T_n}}-\sqrt{\frac{T_n}{T_n+2U_n}}\Big)(p_n p_{-n}-q_n q_{-n})-2\Big].$$

Taking the expectation value, we then get

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}=\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{2m}\sum_n\Big\{\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}\Big)M_n$$

$$+\frac{1}{2}\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)\Big\}\frac{|n|}{2}; \qquad (6.13)$$

in particular, for the lowest state

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}=\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{8m}\sum_n\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)|n|. \qquad (6.13')$$

We must now determine the value of this mean excitation energy below which our method works. In order to find this critical value, we consider a special state in which all particles in the levels between $(3/4)n_{\max}$ and $n_{\max}$ are raised to the levels between $n_{\max}$ and $(5/4)n_{\max}$. In this state the mean excitation energy per particle is $\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{m}\frac{n_{\max}^2}{4^2}$. If the mean excitation energy is larger than this amount in some state, it is certain that in this state some particles are excited to levels higher than $(5/4)n_{\max}$ or some holes are present in levels below $(3/4)n_{\max}$. So our method cannot work consistently in such a case. We find in this way that the necessary condition for the applicability of our method is

$$\frac{1}{N}\langle\mathfrak{H}\rangle_{Av}<\Big(\frac{2\pi\hbar}{L}\Big)^2\frac{1}{m}\frac{n_{\max}^2}{16}. \qquad (6.14)$$

In the lowest state (6.14) is

$$\sum_n\Big(\sqrt{\frac{T_n+2U_n}{T_n}}+\sqrt{\frac{T_n}{T_n+2U_n}}-2\Big)|n|<\frac{n_{\max}^2}{2}. \qquad (6.15)$$

Of course the condition (6.14) or (6.15) is only necessary but not sufficient. There may be cases in which this condition is satisfied and still our method does not work. But such circumstances will occur rather exceptionally.

The condition (6.15) is certainly satisfied if the inter-particle force is so small that we can expand the square root of the form $\sqrt{T_n+2U_n}$ as a power series in U_n and use only a few terms. This condition is just the one required by the validity of the perturbation theory. But the condition (6.15) is weaker than the condition arising in the perturbation theory, because (6.15) does not require that U_n be uniformly small for all values of n. It requires only the smallness of the sum as a whole, and therefore our method has a wider domain of applicability than the perturbation theory. Because of the factor n outside the bracket, terms with n small do not contribute much to the sum on the left-hand side of (6.15), so that J_n with small n may even be very large, provided that T_n+2U_n does not become negative. This fact means that our method is especially suitable for long-range repulsive interactions. As has been first noticed and worked out by Bohm, the method of sound waves provides a very promising means for the study of an assembly of electrons interacting with each other through a repulsive Coulomb force.

§ 7. Concluding remarks.

In concluding the paper, we shall give a few disconnected remarks.

(I) Plasma-like oscillations.

In order to relate the results of our considerations to the work of Bohm, we shall mention briefly the bearing of our results on plasma-like oscillations of a degenerate electron gas. In this case the inter-particle force is the Coulomb repulsion. By Coulomb force is meant here the fact that J_n is proportional to to $1/n^2$. (Notice that the Poisson equation for a point source is $n^2 J_n=\text{const.}$ when expressed in terms of the Fourier transform of the potential). According to (5.12′) we find that in this case V_n is independent of n for small n, i. e. for n so small that we can neglect $\left(\frac{2\hbar\pi}{L}\right)^2 \frac{n_{\max}}{m}$ as compared with $2J_n=\text{const}/n^2$. This means that the frequency of the wave is independent of the wave length, which is characteristic of plasma oscillations.

Another remarkable fact which occurs in the case of Coulomb repulsion is the following. According to (6.11) we find that the matrix elements of the external perturbing force become very small for small n. This means that the plasma oscillations are disturbed very little by external perturbing forces. This fact has been anticipated by Bohm in his paper about superconductivity.[7]

(II) Relation between the two kinds of descriptions.

As we have seen, the system can be described either as an assembly of Fermi particles, or as an assembly of sound quanta. Then the question arises: What relation will exist between the states, in one of which the occupation numbers of the Fermi particles have some specified values, and in the other of which the occupation numbers of the sound quanta have some specified values.[8] As the operator $\frac{1}{2}(p_n^2+q_n^2)-\frac{1}{2}$, which represents the number of sound quanta, and the operator $\psi_n^* \psi_n$, which represents the number of Fermi particles, do not commute with each other, it is clear that one cannot assign numerical values simultaneously to both kinds of occupation numbers. In other words, a state in which the occupation numbers of sound quanta have some definite values is a complicated superposition of various states, each of which is specified by a different set of which is specified by a different set of values of occupation numbers of Fermi particles, and *vice versa*. The statistical relation between the two kinds of occupation numbers will be obtained if one can determine the amplitude of each state in this superposition. We shall describe here briefly the general prescription to get this statistical relation.

First consider the case of a system of non-interacting particles. In this case, we can use $\frac{1}{2}(P_n^2+Q_n^2)-\frac{1}{2}$ instead of $\frac{1}{2}(p_n^2+q_n^2)-\frac{1}{2}$. The operator $\mathfrak{H}$, representing the energy of the system, is a function of the number of sound quanta. At the same time, it is a function of the number of the Fermi particles too. This means that a state in which the occupation numbers of sound quanta have some specified values, and a states in which the occupation numbers of Fermi particles have some specified values, are both some eigen-states of the energy operator. This fact results in all states in the superposition mentioned above belonging to the same eigenvalue of the energy. Now, because the number of linearly independent states belonging to one eigenvalue of $\mathfrak{H}$ is finite, one has only to solve a secular equation of finite degree in order to determine the coefficients in the superposition under consideration. So it is always possible, at least in principle, to answer the question about the statistical relation between the two kinds of occupation numbers.

In the case of interacting particles, we must first express the state in which the occupation numbers of the sound quanta have definite values as a superposition of various states, each having some definite set of values of $\frac{1}{2}(P_n^2+Q_n^2)-\frac{1}{2}$. The problem is none other than to expand Hermite functions of q's in terms of other Hermite functions of Q's. We then apply the method mentioned above to each term of this superposition. In this way we find the required statistical relation for the general case.

In a stationary state the occupation numbers of the sound quanta have some definite values, but the occupation numbers of the Fermi particles do not. The stationary state is a very complicated superposition of states with various numbers of holes and excited particles at various levels. This means that the stationary state is very far from a state where only one particle is excited to some higher level. This fact corresponds to Bohr's statement[9] that the one-particle model is a very bad approximation for the stationary states of such an assembly.

(III) A possible application to the case of exchange forces.

Though our method is applicable only to the case of ordinary forces, there see. to be a hope of applying it also to the case of exchange forces. It is to combine it with Ritz's procedure using a trial function whose form is suggested by our method when applied to ordinary forces. This procedure means physically to replace the exchange force by an equivalent ordinary force in such a way as to gives the best approximation.

Unfortunately, the mathematical structure of the method when generalized to the three-dimensional case is not simple and the author has not yet accomplished it. So in this paper we must be content only to present considerations of a rather mathematical nature without entering into real physical problems.

Acknowledgement.

I would like to express my sincere thanks to Professor J. R. Oppenheimer for giving me the opportunity of working at the Institure for Advanced Study and for his kind interest in this work. I am also indebted to Professor D. Bohm for telling me his interesting results before publication and for fruitful conversation.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

High Energy Nuclear Events

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(Received June 30, 1950)

Abstract

A statistical method for computing high energy collisions of protons with multiple production of particles is discussed. The method consists in assuming that as a result of fairly strong interactions between nucleons and mesons the probabilities of formation of the various possible numbers of particles are determined essentially by the statistical weights of the various possibilities.

1. Introduction.

The meson theory has been a dominant factor in the development of physics since it was announced fifteen years ago by Yukawa. One of its outstanding achievements has been the prediction that mesons should be produced in high energy nuclear collisions. At relatively low energies only one meson can be emitted. At higher energies multiple emission becomes possible.

In this paper an attempt will be made to develop a crude theoretical approach for calculating the outcome of nuclear collisions with very great energy. In particular, phenomena in which two colliding nucleons may give rise to several π-mesons, briefly called hereafter pions, and perhaps also to some anti-nucleons, will be discussed.

In treating this type of processes the conventional perturbation theory solution of the production and destruction of pions breaks down entirely. Indeed, the large value of the interaction constant leads quite commonly to situations in which higher approximations yield larger results than do lower approximations. For this reason it is proposed to explore the possibilities of a method that makes use of this fact. The general idea is the following:

When two nucleons collide with very great energy in their center of mass system this energy will be suddenly released in a small volume surrounding the two nucleons. We may think pictorially of the event as of a collision in which the nucleons with their surrounding retinue of pions hit against each other so that all the portion of space occupied by the nucleons and by their surrounding pion field will be suddenly loaded with a very great amount of energy. Since the interactions of the pion field are strong we may expect that rapidly this energy will be distributed among the various degrees of freedom present in this volume according to statistical laws. One can then compute statistically the probability that in this tiny volume a certain number of pions will be created with a given energy distribution. It is then assumed that the concentration of energy will rapidly dissolve and that the particles into which the energy has been converted will fly out in all directions.

It is realized that this description of the phenomenon is probably as extreme, although in the opposite direction, as is the perturbation theory approach. On the other hand, it might be helpful to explore a theory that deviates from the unknown truth in the opposite direction from that of the conventional theory. It may then be possible to bracket the correct state of fact in between the two theories. One might also make a case that a theory of the kind here proposed may perhaps be a fairly good approximation to actual events at very high energy, since then the number of possible states of the given energy is large and the probability of establishing a state to its average statistical strength will be increased by the very many ways to arrive at the state in question.

The statement that we expect some sort of statistical equilibrium should be qualified as follows. First of all there are conservation laws of charge and of momentum that evidently must be fulfilled. One might expect further that only those states that are easily reachable from the initial state may actually attain statistical equilibrium. So, for example, radiative phenomena in which photons could be created will certainly not have time to develop. The only type of transitions that are believed to be fast enough are the transitions of the Yukawa theory. A succession of such transitions starting with two colliding nucleons may lead only to the formations of a number of charged or neutral pions and also presumably of nucleon-anti-nucleon pairs. The discussion shall be limited, therefore, to these particles only. Notice the additional conservation law for the difference of the numbers of the nucleons and the anti-nucleons.

The proposed theory has some resemblance to a point of view that has been adopted by Heisenberg[1] who describes a very high energy collision of two nucleons by assuming that the pion "fluid" surrounding the nucleons is set in some sort of turbulent motion by the impact energy. He uses qualitative ideas of turbulence in order to estimate the distribution of energy of this turbulent motion among eddies of different sizes. Turbulence represents the beginning of an approach to thermal equilibrium of a fluid. It describes the spreading of the energy of motion to the many states of larger and larger wave number. One might say, therefore, in a qualitative way that the present proposal consists in pushing the Heisenberg point of view to its extreme consequences of actually reaching statistical equilibrium.

The multiple meson production has also been investigated in an interesting paper by Lewis, Oppenheimer and Wouthuysen[2]. These authors stress the importance of the strong coupling expected in the pseudoscalar meson theory for the production of processes of high multiplicity.

In the theory here proposed there is only one adjustable parameter, the volume Ω, into which the energy of the two colliding nucleons is dumped. Since the pion field surrounding the nucleons extends to a distance of the order $\hbar/\mu c$ where μ is the pion mass, Ω is expected to have linear dimensions of this order of magnitude. As long as the Lorentz contraction is neglected one could take for example a sphere of radius $\hbar/\mu c$. However, when the two nucleons approach each other with very high energy in the center of gravity system, their surrounding pion clouds will be Lorentz contracted and the volume will be correspondingly reduced.

For this reason the volume Ω will be taken energy dependent according to the relationship:

$$\Omega=\Omega_0\frac{2Mc^2}{W}, \tag{1}$$

where Ω_0 is the volume without Lorentz contraction. W is the total energy of the two colliding nucleons in the center of gravity system and M is the nucleon mass. The factor $2Mc^2/W$ is the Lorentz contraction. The uncontracted volume Ω_0 may be taken as a sphere of radius R:

$$\Omega_0=4\pi R^3/3. \tag{2}$$

It is found in the applications that one seems to get an acceptable agreement with known facts by assuming:

$$R=\hbar/\mu c=1.4\times 10^{-13}\text{cm}. \tag{3}$$

This choice of the volume, although plausible as order of magnitude, is clearly arbitrary and could be changed in order to improve the agreement with experiment. One finds that an increase of Ω_0 would tend to favor processes in which a large number of particles is created.

According to this point of view the total collision cross-section of the two nucleons will be always of the order of magnitude of the geometrical cross-section of the pion cloud. In the numerical calculations actually, the total cross-section has been taken equal to area of a circle of radius R, namely,

$$\sigma_{tot}=\pi R^2. \tag{4}$$

Assuming (3) one finds $\sigma_{tot}=6\times 10^{-26}\text{cm}^2$. In order to compute the partial cross-section for a phenomenon in which for example three pions are produced in the collision, one will multiply the total cross-section (4) by the relative probability that three pions instead of any other possible number and kind of particles are produced.

The probability of transition into a state of a given type is proportional to the square of the corresponding effective matrix element and to the density of states per unit energy interval. Our assumption of a statistical equilibrium consists in postulating that the square of the effective matrix element is merely proportional to the probability that, for the state in question, all particles are contained at the same time inside Ω. For example in the case of a state that describes n completely independent particles with momenta $p_1, p_2, \cdots p_n$ this probability is $(\Omega/V)^n$ where V is the large normalization volume. The number of states per unit energy interval is

$$\left(\frac{V}{8\pi^3\hbar^3}\right)^n \frac{d}{dW} Q(W),$$

where $Q(W)$ is the volume of momentum space corresponding to the total energy W. The probability for the formation of the state in question is therefore assumed to be proportional to the product:

$$S(n) = \left(\frac{\Omega}{8\pi^3\hbar^3}\right)^n \frac{dQ(W)}{dW}. \tag{5}$$

There are some complications arising from the fact that the particles are not independent.

a) In the center of mass system the positions and momenta of only $n-1$ of the n particles are independent variables. For this reason the exponent of Ω will be $n-1$ instead of n. Also the momentum space $Q(W)$ will be $3(n-1)$-dimensional instead of $3n$-dimensional.

b) Some of the particles may be identical and this fact should be taken into account in computing $Q(W)$.

c) Some of the particles may carry a spin and one should then allow for the corresponding multiplicity of the states.

d) The conservation of angular momentum restricts the statistical equilibrium to states with angular momentum equal to that of the two colliding nucleons. In all cases considered $\lambda\!\!\!^{-}$ for the nucleons is smaller than the radius $\hbar/\mu c$ of the sphere of action. It is then meaningful to discuss separately collisions with various values of the impact parameter b (b is the distance of the two straight lines along which the nucleons move before the collision). In units of $\hbar$ the angular momentum is $l = b/\lambda\!\!\!^{-}$. The cross-section for collisions with impact parameter between b and $b+db$ is $2\pi b db = 2\pi \lambda\!\!\!^{-2} l dl$. One should treat separately collisions with different values of the impact parameter and compute for each of them the probability of the various possible events. The cross-section for a special event is then obtained by adding the contribution of the various l-values.

It is found in most cases that the results so obtained differ only by small numerical factors from those obtained by neglecting the conservation of angular momentum.

This has been done as a rule in order to simplify the mathematics. The corrections arising from the conservation of angular momentum have been, however, indicated in typical cases.

2. Example. Pion Production in Low Energy Nucleon Collisions.

As a first example the production of pions in a collision of two nucleons with relative energy barely above the threshold needed for emission of a pion will be discussed. This example is chosen because it is the simplest possible. It is, however, a case in which the statistical approach may be misleading, since only few states of rather low energy are involved. We will first simplify this example by disregarding the spin of the nucleons as well as the possible existence of a spin of the pions and by disregarding also the various possible electric charges of the particles in question. In the center of gravity system we will have therefore two nucleons colliding against each other. $T/2$ is the kinetic energy of each of the two nucleons. A pion can be emitted when $T > \mu c^2$. We shall assume that this inequality is fulfilled; that the kinetic energy, however, exceeds the threshold by only a small amount, so that both the two nucleons and the pion that may be formed will have non-relativistic energies.

Conservation of energy in this case allows only two types of states; Type (a) in which the two nucleons are scattered elastically without formation of pions; and type (b) in which a pion is formed and three particles, two nucleons and a pion, emerge after the collision.

The statistical weight of the states of type (a) is obtained as follows. Since the momenta of the two nucleons are equal and opposite the momentum space will be three-dimensional. We can compute the statistical weight with (5). s will be taken $=1$ because the momentum of one particle determines that of the other. The reduced mass is $M/2$, the momentum $p=\sqrt{MT}$ and the phase space volume $Q(T)=4\pi p^3/3$. According to (5) the statistical weight of this state has therefore the familiar expression:

$$S_2=\frac{\Omega M^{3/2}}{4\pi^2\hbar^3}\sqrt{T}\,. \tag{6}$$

S_2 should be compared with the statistical weight S_3 for case (b) in which three particles, two nucleons and one pion, emerge. Since the total momentum is zero, only the momenta of two of the three particles are independent and therefore in (5) $s=2$. The calculation of the momentum volume involves some slight complication on account of the conservation of momentum. Let p be the momentum of the pion and let the momenta of the two nucleons be $-\frac{1}{2}p\pm q$. The kinetic energy will then be:

$$T_1=\left(\frac{1}{2\mu}+\frac{1}{4M}\right)p^2+\frac{1}{M}q^2 \tag{7}$$

where $T_1=T-\mu c^2$ is the kinetic energy left over after a pion has been formed. Formula (7) represents an ellipsoid in the six dimensional momentum space of the two vectors p and q. Its volume is:

$$Q_3=\frac{\pi^3}{3!}\left(\frac{4M^2\mu}{2M+\mu}\right)^{3/2}T_1^3. \tag{8}$$

The factor $\pi^3/3!$ is, for a six-dimensional sphere, the analog of the factor $4\pi/3$ in the volume of an ordinary sphere. Substituting in (5) one finds:

$$S_3=\frac{\Omega^2}{16\pi^3\hbar^6}\left(\frac{M^2\mu}{2M+\mu}\right)^{3/2}T_1^2\approx\frac{\Omega^2M^{3/2}\mu^{3/2}T_1^2}{32\sqrt{2}\,\pi^3\hbar^6}. \tag{9}$$

The last expression is simplified by assuming $\mu \ll M$. The probabilities of the two events (a) and (b) are proportional to S_2 and S_3. Since S_3 is very small, we may take the ratio S_3/S_2 to be the probability that the collision leads to pion formation. This is given by:

$$\frac{S_3}{S_2}=\frac{\Omega\mu^{3/2}}{8\sqrt{2}\,\pi\hbar^3}\,\frac{(T-\mu c^2)^2}{\sqrt{T}}\approx\frac{\Omega\mu(T-\mu c^2)^2}{8\sqrt{2}\,\pi\hbar^3c}.$$

Since T is barely larger than the threshold energy μc^2, this value has been substituted for T in the denominator. In this case also the Lorentz contraction of the two colliding nucleons is negligible and we can therefore substitute for Ω the value Ω_0 given by (2) and (3). One finds:

$$\frac{S_3}{S_2}=\frac{1}{6\sqrt{2}}\left(\frac{T}{\mu c^2}-1\right)^2. \tag{11}$$

The cross-section for pion formation is given by the product of the total cross-section (4) and the probability (11). For example, in a bombardment of nucleons at rest with 345 MeV nucleons, (the proton energy available at Berkeley) one finds that the energy available in the center of gravity system is $T=165$MeV. On the other hand $\mu c^2=140$MeV and the previous formula gives therefore $S_3/S_2=.0038$. This means that at this bombarding energy a pion will be formed in about 0.4 per cent of the nucleon collisions.

If one examines the process more in detail one will recognize that in a collision of two protons the probability of emission of a positive pion is twice (11) namely, .0076. Because if a positive pion is formed a proton and a neutron, instead of two protons, will also emerge. Their statistical weight is twice that for two protons because they are not identical particles. Similarly in the collision of a proton and a neutron the probability of emission of a positive pion is one-half of (10) namely, .0019. The probability of emission of a negative pion is the same.

For example, when a carbon target is bombarded by 345 MeV protons the probabilities that the collision takes place between the proton and another proton or a neutron are the same. Hence, the probability of emission of a positive pion will be $.0076/2+.0019/2$; that is, .0048; and the probability of emission of a negative pion will be $.0019/2=.001$. Since the nuclear cross-section of carbon is about 3×10^{-25}cm^2, one will obtain the expected values of the cross-sections for emission of a positive and a negative pion by multiplying the nuclear cross-section by the above probabilities. The results are $1.4 \times 10^{-27}\text{cm}^2$ for the positive and $3 \times 10^{-28}\text{cm}^2$ for the negative pions. Considering the extremely crude calculation these values are in surprisingly good agreement with the experimental results.

In the above discussion the conservation of angular momentum has been disregarded. When a pion is produced the kinetic energy of the three emerging particles is small and they will therefore escape in an s-state. Consequently, only the initial states of zero angular momentum can be contribute to this type of final state. Their maximum cross-section has the well known expression $\pi\bar{\lambda}^2$ which is appreciably smaller than (4). However, also the competition of elastic scattering versus pion production is less since only the scattering states of zero angular momentum will contribute.

By carrying out the calculation one finds that the two effects almost cancel each other and that the conservation of angular momentum changes the previous results for the cross-section for pion production by only a factor 2/3. As long as the conservation of angular momentum is neglected one expects the scattering of the two nucleons to be spherically symmetrical in the center of mass system. This is no longer the case when the angular momentum is conserved. One finds then that the elastic scattering cross-section per steradian in the center of mass system instead of being constant is approximately proportional to $1/\sin\theta$, where θ is the scattering angle in the same system.

3. Formulas for the Statistical Weights.

Some standard formulas expressing the statistical weights, S, for a number of simple cases will be collected here.

First, the case will be considered that after a collision n particles emerge with masses m_1, $m_2 \cdots\cdots m_n$. Neglecting spin properties and assuming that the particles are statistically independent, and disregarding also the momentum conservation, one finds for S the following two formulas corresponding to the classical and to the extreme relativistic case :

$$S_n = \frac{(m_1 m_2 \cdots m_n)^{3/2} \Omega^n}{2^{3n/2} \pi^{3n/2} \hbar^{3n}} \frac{T^{3n/2-1}}{(3n/2-1)\,!}, \tag{12}$$

(classical case)

$$S_n = \frac{\Omega^n}{\pi^{2n} \hbar^{3n} c^{3n}} \frac{W^{3n-1}}{(3n-1)\,!}. \tag{13}$$

(extr. relativistic case)

In (12) T is the total classical kinetic energy of the n particles and in (13) W is the total energy including rest energy of the n particles. One can also compute a formula for the case that s of the particles, usually the nucleons, are classical and n, usually the pions, are extreme relativistic. Neglecting again spin statistics and momentum conservation and assuming further that all the classical particles have the nucleon mass, M, one finds:

$$S(s,n)=\frac{M^{3s/2}\Omega^{n+s}}{2^{3s/2}\pi^{2n+3s/2}\hbar^{3s+3n}c^{3n}}\,\frac{(W-sMc^2)^{3n+3s/2-1}}{(3n+3s/2-1)!}. \tag{14}$$

It is sometimes convenient to re-write (14) in the following form:

$$S(s,n)=\frac{M^{3s/2}\Omega^{s/2+1/3}}{2^{3s/2}\pi^{s/2+2/3}\hbar^{3s/2+1}c^{-3s/2+1}}\,\frac{\left(\dfrac{\Omega^{1/3}(W-sMc^2)}{\pi^{2/3}\hbar c}\right)^{3n+3s/2-1}}{(3n+3s/2-1)!}, \tag{15}$$

since it is thus easy to obtain an approximate expression for the sum of the statistical weights $S(s,n)$ over all values of n. The approximation applies to the cases when the average value of n is $\gg 1$. One finds then:

$$\sum_{n=0}^{\infty} S(s,n)\approx\frac{M^{3s/2}\Omega^{s/2+1/3}}{3\times 2^{3s/2}\pi^{s/2+2/3}\hbar^{3s/2+1}\,c^{-3s/2+1}}\exp\left(\frac{\Omega^{1/3}(W-sMc^2)}{\pi^{2/3}\hbar c}\right). \tag{16}$$

The numerical values of (15) and (16) adopting for Ω (1), (2) and (3) are:

$$Mc^2S(s,n)=\frac{6.31}{w^{1/3}}\left(\frac{98.8}{w}\right)^{s/2}\frac{\left(6.31\,\dfrac{w-s}{w^{1/3}}\right)^{3n+3s/2-1}}{(3n+3s/2-1)!} \tag{17}$$

and:

$$Mc^2\sum_{n=0}^{\infty} S(s,n)\approx\frac{2.10}{w^{1/3}}\left(\frac{98.8}{w}\right)^{s/2}\exp\left(6.31\,\frac{w-s}{w^{1/3}}\right) \tag{18}$$

where one has put

$$w=W/Mc^2 \tag{19}$$

and it has further been assumed $\mu/M=.15$.

In the previous formulas the momentum conservation has been disregarded. The formulas, however, can be generalized without difficulty so as to introduce at least approximately the requirement that the total momentum be zero. The approximation consists in assuming that the mass of the pion is very small compared to the nucleon mass. The momentum of the nucleons will then be much greater than the momentum of the pions since the kinetic energy is approximately equipartitioned among the various particles. Therefore, one can approximately apply the condition that the sum of the momenta vanishes to the nucleons only. One recognizes then that formula (15) must be changed as follows. a) Instead of s one will write $s-1$ at all places except in the term $W-sMc^2$ since we have now $s-1$ independent momenta of the heavy particles. b) The factor $M^{3s/2}$ must be changed because instead of the mass one should substitute an expression which is the analog of a reduced mass. It is found that the factor in question must be substituted by $M^{3(s-1)/2}/s^{3/2}$. When the conservation of momentum is approximately taken into account formulas (17) and (18) should then be changed as follows:

$$Mc^2 S(s, n) = \frac{6.31}{s^{3/2} w^{1/3}} \left(\frac{98.8}{w} \right)^{(s-1)/2} \frac{(6.31(w-s)/w^{1/3})^{3n+3s/2-1}}{(3n+3s/2-1)!}, \tag{20}$$

$$Mc^2 \sum_{n=0}^{\infty} S(s, n) \approx \frac{2.10}{s^{3/2} w^{1/3}} \left(\frac{98.8}{w} \right)^{(s-1)/2} \exp(6.31(w-s)/w^{1/3}). \tag{21}$$

In all the preceding formulas the particles have been assumed to be statistically independent. As long as few nucleons and pions are involved, the error is not great. Larger errors are expected for high multiplicity. The formulas, however, become quite involved since there are at least three kinds of pions and four kinds of nucleons and anti-nucleons. No attempt has been made to introduce these complications for phenomena of relatively low energy. They have been calculated as if there were only one type of pions, one of nucleons and one of anti-nucleons statistically independent. This procedure is certainly inadequate and will give a too high multiplicity at high energy. For phenomena of extremely high energy it becomes simple to introduce the statistical correlations by substituting the statistical by a thermo-dynamical model. This case will be treated in Section 6.

In the previous expressions also the conservation of angular momentum has been neglected. The error introduced with this omission will be discussed in Section 6, where an appropriate correction factor for it will be given.

4. Transition from Single to Multiple Production of Pions.

In Section 2 the emission of a single low energy pion has been discussed. Collisions of higher energy in which besides the two original nucleons also several pions may be produced will be considered now. A rough indication of the features of this process may be obtained by computing the relative probabilities for the emission of 0, 1, 2,……n… pions with (20). In that formula one will put $s=2$. Statistical correlations and conservation of angular momentum will be negelcted. Omitting a common factor, the probabilities of the various values of n are proportional to:

$$\left\{ \frac{251}{w} (w-2)^3 \right\}^n \Big/ \left(\frac{3}{2} \times \frac{5}{2} \times \cdots \times \frac{6n+1}{2} \right). \tag{22}$$

Table I gives the probabilities of pion production of different multiplicities calculated according to this formula. The first column of the table gives the energy, w, of the two nucleons in the center of gravity system in units of Mc^2. The second column gives the energy, w', of the primary particle in the laboratory frame of reference. The next eight columns are labeled by the number, n, of pions produced and give the probabilities of various events in per cents. The last column gives the average number of pions produced.

TABLE I

w	w'	$n=0$	1	2	3	4	5	6	7	$\bar{n}$
2.5	2.1	49	47	4						.6
3	3.5	9	59	30	2					1.2
3.5	5.1	2	31	46	18	3				1.9
4	7.0		13	40	33	11	2			2.5
5	11.5		2	15	34	31	14	3	1	3.5

Notice that already for a bombarding energy of about 1 BeV corresponding to the first line of the table the probability of elastic collision of the two nucleons is 50%. This probability decreases rapidly and drops below one per cent for bombarding energies of about 5 BeV. As the bombarding energy increases the probability of multiple phenomena increases as indicated in the table. The most probable value of n according to (22) should be given approximately by $2.1(w-2)/w^{1/3}$.

It will be seen in Section 6 that at high energy very appreciable errors are are introduced by neglecting the angular momentum conservation and the statistical correlations. Table I gives only a qualitative indication of the transition from elastic scattering to single and then multiple pion production. The quantitative features of the multiple production, however, should be more reliably represented by formula (32).

5. Production of Anti-Nucleons.

When the two colliding pions have a total energy $> 4Mc^2$ in the center of gravity system, competition with processes in which a nucleon-anti-nucleon pair is formed becomes possible. When the energy is barely above the $4Mc^2$-threshold, no pions can be formed accompanying the pair. As the energy increases, however, the pair will be as a rule accompanied by a number of pions. For moderate energy $w < 10$ one will use formula (20). Substituting in it $s=4$ we obtain the statistical weight for nucleon pair formation associated with the emission of n pions. Substituting $s=2$ we obtain an expression proportional to the probability that no pair is formed and the two original nucleons plus n pions emerge.

Omitting the common factor Mc^2 one obtains from (20) :

$$S(4, n) = \frac{775}{w^{11/6}} \frac{(6.31(w-4)/w^{1/3})^{3n+7/2}}{(3n+7/2)!}. \tag{23}$$

In normalizing these probabilities to total probabilitiy$=1$, one can make use of the fact that the probability of pair formation in the range of energies here discussed is always less than one per cent. One can therefore disregard the pair formation in the normalization factor which reduces to $\sum_n S(2, n)$. In calculating this sum one can use (21). One obtains in the end the following expression for the probability of pair formation accompanied by n pions :

$$P(4, n) = \frac{105}{w}\left(6.31\,\frac{w-4}{w^{1/3}}\right)^{3n+7/2} \frac{\exp(-6.31(w-2)/w^{1/3})}{(3n+7/2)\,!}.$$

TABLE II

w	w'	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	Total
$4+\varepsilon$	7.0	1000 $\varepsilon^{7/2}$					$.1\varepsilon^{7/2}$
4.5	9.1	14	.6				15×10^{-4}
5	11.5	27	8	.7			36×10^{-4}
5.5	14.3	21	21	5	.5		47×10^{-4}
6	16.9	12	25	14	3	.3	54×10^{-4}

Columns 3 to 7 inclusive $\times\ 10^{-4}$

Table II is calculated with this formula. Again the first and second columns represent in units of Mc^2 the total energy in the center of gravity system and the total energy of the bombarding particle in the laboratory system. The next five columns give the probabilities of pair formation accompanied by n pions. These probabilities have been multiplied by a factor 10^4. The eighth column is the total probability of pair formation.

Again, in computing this table the statistical correlations mentioned in Section 3 and the conservation of angular momentum have been disregarded. For this reason the data of the table are merely indicative of the results that would be given by a more correct computation.

At the highest energy here considered the probability of anti-nucleon formation is 0.005. Since in a collision of this energy probably two or three pions are formed in the average one concludes that at these energies the ratio of anti-nucleons to pions formed is about 0.002. Therefore, anti-nucleons will be hard to find even in fairly high energy collisions.

6. Collisions of Extremely High Energy.

In discussing the collision of two nucleons with extremely high energy one can simplify the calculations by assuming that all the various particles produced are extreme relativistic and that thermo-dynamics may be applied instead of a detailed statistical computation of the probabilities of the various events.

In this discussion the conservation of angular momentum will first be neglected. Its effect will be given at the end of this section.

The extremely high energy density that is suddenly formed in the volume Ω will give rise to multiple production of pions and of pairs of nucleons and anti-nucleons. Since both kinds of particles are extreme relativistic, the energy density will be proportional to the fourth power of the temperature, T, as in Stefan's law.

The pions, like the photons, obey the Bose-Einstein statistics. Since we further assume that the temperature is so high that the rest mass is negligible, their energy momentum relationship will be the same as for the photons. Consequently the Stefan's law for the pions will be quite similar to the ordinary Stefan's law of the black body radiation. The difference is only in a statistical weight factor. For the photons the statistical weight is the factor, 2, because of the two polarization directions. If we assume that the pions have spin zero and differ only by their charge $\pm e$ or 0, their statistical weight will be 3. Consequently, the energy density of the pions will be obtained by multiplying the energy density of the ordinary Stefan's law by the factor 3/2. This energy density is therefore:

$$\frac{3\times 6.494\,(kT)^4}{2\pi^2\hbar^3c^3}. \tag{25}$$

The numerical factor $6.494=\pi^4/15$ is six times the sum of the inverse fourth powers of the integral numbers.

The contribution of the nucleons and anti-nucleons to the energy density is given by a similar formula. The differences are that the statistical weight of the nucleons is eight since we have four different types of nucleons and anti-nucleons and for each, two spin orientations. A further difference is due to the fact that these particles obey the Pauli principle. In the extreme relativistic case their energy density is:

$$\frac{4\times 5.682\,(kT)^4}{\pi^2\hbar^3c^3}. \tag{26}$$

Here the numerical factor 5.682 is $6\sum_1^\infty(-1)^{n+1}/n^4$.

The temperature is obtained by equating the total energy to the product of the volume Ω times the sum of the two energy densities (25) and (26). Making use of (1) one obtains the temperature from the following equation:

$$(kT)^4=.152\frac{\hbar^3c^3W^2}{Mc^2\Omega_0}. \tag{27}$$

In order to compute the number of pions, nucleons and anti-nucleons produced we need formulas for the density of the various particles. These are computed according to standard procedures of statistical mechanics. In the extreme relativistic case the density of the particles turns out to be proportional to the third power of the temperature. The total densities of the pions and of the nucleons are given by the following two expressions:

$$n_\pi=.367\frac{(kT)^3}{\hbar^3c^3}; \qquad n_N=.855\frac{(kT)^3}{\hbar^3c^3}. \tag{28}$$

The total numbers of pions and nucleons are obtained by multiplying the expressions (28) by the volume Ω and by substituting in them the temperature calculated from (27). The result must be finally corrected in order to take into account the conservation of angular momentum. Only the result of this correction will be given. It is found that conservation of angular momentum has the effect of reducing the numbers of pions and nucleons by a factor that has been calculated numerically to be about .51. The conservation of angular momentum has the further effect that the angular distribution of particles produced is no longer isotropical but tends to favor somewhat, particles moving parallel to the original direction of the two colliding nucleons. Introducing these corrections one finds that the number of pions is:

$$\text{No. of pions} = .091\left(\frac{\Omega_0 M W^2}{c\hbar^3}\right)^{1/4} = .54\sqrt{W/Mc^2} \tag{29}$$

and the number of nucleons plus anti-nucleons is:

$$\text{No. of nucleons and anti-nucleons} = .21\left(\frac{\Omega_0 M W^2}{c\hbar^3}\right)^{1/4} = 1.3\sqrt{W/Mc^2}\,. \tag{30}$$

From this follows that the number of charged particles that emerge out of an extremely high energy collision is given by:

$$1.2(W'/Mc^2)^{1/4}. \tag{30a}$$

(W' = energy in the laboratory system)

In these tormulas Ω_0 has been substituted by its value, (2), (3).

These formulas apply only to extreme high energies. Substituting the value, (2), (3), for Ω_0 one finds from (27) that the relationship between temperature and energy can be written in the form:

$$kT/Mc^2 = .105\sqrt{W/Mc^2}\,. \tag{31}$$

Relativistic conditions for the nucleons will be achieved therefore only when $W > 100Mc^2$. This corresponds in the laboratory system to an energy of the bombarding particles of more than 5×10^{12} eV. At somewhat lower energies the number of anti-nucleon pairs formed will decrease very rapidly, especially since an energy $2Mc^2$ is needed in order to form a pair. In this energy range the formation of paris is probably better represented by the computation of Section 5.

A comparison of (29) and (30) indicates that in such collisions of extremely high energy the number of nucleons and anti-nucleons produced exceeds that of the pions. Naturally, the anti-protons which are the particles in which we are most interested from the experimental point of view are only one-fourth of the particles (30). Therefore, a somewhat larger number of pions than of anti-protons is formed, even at these high energies. The reason why so many nucleons of all kinds are formed compared to the pions is their statistical weight (8 for the nucleons, 3 for the pions).

In an intermediate energy range where the multiple production of pions is the relevant phenomenon one can still apply the thermo-dynamic method restricting, however, the thermo-dynamic equilibrium to the pion gas only, and assuming that the activation energy of the pairs is too high for producing a sizeable number of these particles at the given temperature. The energy density in this case will be given by (25). The numerical coefficient in formula (27) will be reduced for this reason from .152 to .046. Also in the same formula one will substitute W by $W-2Mc^2$ since the energy of the two nucleons does not contribute to the energy of the pion gas. Introducing also the factor .51 for the conservation of angular momentum one finds that the number of pions in this approximation is given by:

$$\text{No. of pions} = .323\frac{M^{1/4}R^{3/4}(W-2Mc^2)^{3/2}}{\hbar^{3/4}c^{1/4}W} = 1.34\frac{(w-2)^{3/2}}{w}, \tag{32}$$

where $w = W/Mc^2$.
In the intermediate energy range of bombarding particles from 10 to 100 BeV this formula probably gives a better estimate of the multiplicities than do the computations of Table I. In particular it would appear that especially the multiplicities given in the last two lines of Table I are too large. According to (32) one would expect for these two energies multiplicities of about 2 instead of the considerably higher values given in Table I. The difference is due to two effects which have been disregarded in computing Table I; namely, the statistical correlation between various types of pions and the angular momentum conservation. Both factors are approximately taken into account in formula (32).

Since no observation of multiple production of an isolated nucleon is available at present, the comparison of these findings with experimental results is only tentative. The present theory seems to give rather low multiplicities except at extremely high energies of the order of 10^{12} to 10^{13} eV. As more experimental results become available it may be possible to improve the agreement of the theory with experiment by changing the choice (3) of R. If experimentally the multiplicities should turn out to be larger than according to the theory, one would increase R or decrease it in the opposite case.

In the present theory we have considered only one type of mesons, the pions. If mesons of larger mass strongly bound to nucleons should exist, as seems to be indicated by the recent experiments of Anderson[3], these particles also could reach statistical equilibrium. Since their rest energy is large, however, they would compete unfavorably with the production of pions except at very high energies. One would expect therefore in most collisions that the number of pions produced should be appreciably larger than that of the heavier mesons.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Nuclear Saturation Phenomena deducible from Pair Theories

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(Received July 16, 1950)

Abstract

Previous work on the same topic is extended, especially to include interactions involving the nuclear spins or isotopic spins in the static limit. For the discussion of such theories, a strong coupling approximation is used. The results vary widely with the coupling type assumed, resembling single particle models of nuclear structure in some cases, and liquid drop models, in particular the α-particle model, in other cases.

I. Yukawa Theory and Pair Theory of Nuclear Forces.

The pair theory of nuclear forces has hardly ever been considered a serious rival to Yukawa's theory. Both theories are about equally old.[1] Since the discovery of the π-meson it has become increasingly clear that this particle has basically the properties of a Yukawa particle. While the qualitative evidence thus favors the Yukawa theory, the fundamental difficulties of all quantized field theories render it hard, if not impossible, to decide how closely this theory agrees with reality. It certainly furnishes no clue to the puzzling role played by the μ-meson among the "elementary particles".

According to the μ-pair theory, first proposed by Marshak,[2] two μ-mesons act as carriers of the nuclear interactions, instead of the single π-meson of Yukawa's theory. Actually the two theories can be merged into one, thus providing a unified picture of both meson types and allowing even for the existence of heavier, short-lived, mesons. The starting point of this theory[3] is the elementary interaction law of the pair theory (see eq. (1) below), admitting, for instance, the emission by a nucleon of a pair of μ-meson, one or both of which may be neutral. Such a μ-pair, because of its capability to transform into a nucleon-antinucleon pair as a virtual intermediate state, can be bound together to form a particle of integral spin whose mass is less than that of the two free μ-mesons by a certain amount of binding energy. This composite particle is then interpreted as the π-meson. Indeed, it has just the basic properties of a Yukawa particle, being a scalar, or vector, or pseudo-vector, or pseudo-scalar meson depending on the interaction type assumed in the pair theory.

A nucleon may emit a μ-pair also in this bound state; the matrix elements calculated for this process of π-meson emission coincide with those adopted in various Yukawa theories. The most promising interaction type is the pseudoscalar one (leading to a pseudoscalar π-meson) because it offers a natural explanation for the weakness of the interaction of μ-mesons with slow nucleons, while leaving the strength of the π-meson-nucleon interaction unimpaired. Interactions between two nucleons are transmitted by both free and bound pairs, resulting in a mixture of the force types known from Yukawa and pair theories.

In view of this theoretical possibility, the μ-pair theory may deserve a renewed interest, in particular with regard to such problems where the differences between the two theories are most profound. One such problem is the saturation character of the nuclear forces. In Yukawa's theory, the saturation appears as a consequence of the exchange character of the forces. In pair theories, as was first pointed out by Critchfield and Teller,[4] there is a new mechanism providing for saturation even if no charge or spin exchange takes place. It is this mechanism which shall be further investigated in the following pages. In this particular context the π-meson field is evidently of minor importance, and we propose to disregard it for the sake of simplifying the argument.

II. Review of Previous Work and Outline of Present Objectives.

The most elementary saturation effect, as derived from certain pair theories, is the following:

(A) The potential energy of n nucleons located at the same point or within a very small cell is proportional to $(n-1)$.[4]

As to the allowable size of the cell, it will become apparent from what follows that the source functions assigned to the various nucleons should still overlap almost completely, i.e., the cell radius should be considerably smaller than the source radius R. R plays the part of a fundamental length in these theories, and also determines the range of the nuclear forces.

In addition, we shall discuss a saturation phenomenon of a different, though related, type:

(B) If two small cells containing n_1 and n_2 nucleons respectively, interact over a distance $\gtrsim R$, their mutual potential energy, as a function of n_1, n_2, approaches a constant value as n_1 and n_2 increase beyond a certain critical value. Similarly for three-body forces, etc.

The statements (A) and (B), if both valid, have an interesting implication. Suppose that one nucleon is transferred from the first into the second cell: $n_1 \rightarrow n_1-1$, $n_2 \rightarrow n_2+1$. If both (n_1-1) and n_2 are well above the critical value, this process leaves the energy unchanged, according to (A) and (B), even if other nucleons are present outside. If the whole surrounding region is covered with well filled cells, the particle can travel inside this region without experiencing any forces whatsoever. Its mean free path in such nuclear matter is infinite. Local density fluctuations are irrelevant as long as the population of the cells remains above the critical one. The outside boundary surface of a nucleon acts, of course, as a repulsive potential (because of (A)). Clearly, this situation is strongly reminiscent of the "single particle models" frequently used in the analysis of nuclear phenomena, especially in the "shell model" studies.[5)]

Although of minor importance for our present purposes, a third saturation effect may be mentioned:

(C) In nuclear matter of high uniform density, the energy density is proportional to the particle density.[6)]

For (C) to be valid, the density must be so large that the source functions of neighboring nuclei overlap strongly. Actually, the calculations[6)] were carried out for a lattice arrangement of the nucleons, and the saturation was shown to set in as the lattice constant becomes smaller than the source radius R.

So far, the situation might seem rather satisfactory if it were not for a major objection that must be raised against all the investigations referred to: They are quite incomplete in that they deal only with those simpler versions of the pair theory which, in a static approximation (nucleons infinitely heavy and at rest), involve neither the spins nor the isotopic spins of the nucleons. These theories are mathematically instructive because they allow of rigorous solutions, regardless of the coupling strength, but they obviously fail to explain the spin dependence of the forces. In the other theories, only weak and strong coupling approximations are practicable, and no saturation effects can occur in the weak coupling case. We therefore propose to examine some typical samples of such pair theories in the strong coupling approximation. It is true that this approximation furnishes a much too weak spin dependence of the nuclear forces.[7)] But, extrapolated down to intermediate coupling strengths, which is the case to be considered most seriously, our results will at least indicate for which theories the desired saturation effects may be expected at all, or which modifications may be anticipated. Actually, the strong coupling approximation will reveal a great diversity of behavior, even among the few sample cases selected.

Unfortunately, the pseudoscalar coupling theory which, as mentioned above, seems most promising, leads to a more complicated mathematical problem because the pseudoscalar interaction vanishes, strictly speaking, in the static limit, and a first non-static correction has to be considered. This case shall not be included in this preliminary survey, but it is felt that the results, because of a general resemblance of the problems, will be of a similar nature as those arrived at here.

At all events, the reader will be aware of the highly idealized character of the mathematical problem. Accordingly, the results are to be taken *cum grano salis*. This is also the reason for disregarding entirely such related phenomena as the magnetic moments of the nucleons which require a more realistic approach.

III. Static Approximation: Rigorous Solution.

The interaction Hamiltonian of pair theories has the general structure:

$$H_i=\eta\int d^3x(\phi^*A\phi)(\psi^*a\psi). \tag{1}$$

Here, ϕ and ψ are the nucleonic and mesonic field operators, satisfying well known commutation rules; A and a are appropriate matrix operators, and η is the coupling constant. The static approximation is obtained by the substitution

$$\phi^*A\phi\rightarrow\sum_k A_k\delta(x-x_k) \tag{2}$$

where the suffix k distinguishes the various nucleons with fixed positions x_k.

Actually, to avoid infinities, the relativistic interaction laws (1) have to be modified by the introduction of extended source functions, or an equivalent momentum cut-off. In the static limit, this affects only the meson field. We follow the customary procedure[4),6),7)] to average ψ and ψ^* independently over a spherical region of linear dimensions R:

$$\psi(x_k)\rightarrow\bar\psi(x_k)=\int d^3xU(x-x_k)\psi(x). \tag{3}$$

Together with (2), this changes the interaction into

$$H_i=\eta\sum_k A_k\bar\psi^*(x_k)a\bar\psi(x_k). \tag{4}$$

Now, in order to include situations discussed in Section II, let us assume n_1 nucleons to be located at (or very close to) the point x_1, n_2 nucleons at x_2, and so forth. Since the nucleons of one such cluster give rise to the same (or almost the same) source function $U(x-x_k)$ in (3), it is convenient to distinguish them by a second suffix $l(=1\cdots n_k)$:

$$H_i=\eta\sum_k\left(\sum_{l=1}^{n_k}A_{kl}\right)\bar\psi^*(x_k)a\bar\psi(x_k). \tag{5}$$

Let us first re-examine the more primitive theories in which A_{kl} is the unit operator:

$$H_i=\eta\sum_k n_k\bar\psi(x_k)a\bar\psi(x_k). \tag{6}$$

This is the type of problem referred to in Section II, for which rigorous solutions can be found (by the introduction of normal modes) and for which the saturation properties (A) and (C) have been established. We want to indicate briefly how (B) can be justified.

Changing the occupation number n_k of one cluster is equivalent to changing the coupling parameter of the corresponding term in (6): $\eta_k=\eta n_k$. Previous calculations for the case of single occupation are easily generalized to allow for different η_k values. For two clusters, separated by a distance $r>R$ such that no overlapping of the two source functions occurs:

$$U(x-x_1)U(x-x_2)=0, \tag{7}$$

the situation is as follows.

The simplest example is the scalar pair theory where ψ stands for a scalar complex field, and $a=1$. For $n_1=n_2=1$, this problem has been treated in W, and the generalization is straightforward.* The mutual potential of the two clusters is found to depend on n_1, n_2 in the following fashion.**

$$V(r)=J(r)[(\eta n_1)^{-1}+R^{-1}]^{-1}[(\eta n_2)^{-1}+R^{-1}]^{-1}$$

where $J(r)$ is a distance function independent of n_1 and n_2. V, as a function of each n_k, approaches a constant value as n_k increases beyond the critical value R/η, in accordance with (B). In particular, if $\eta \gg R$, i.e. in the strong coupling case, the critical value is small, and V becomes almost independent of n_1 and n_2, even down to n_1 or $n_2=1$.

Slightly more involved is the case of a spinor field ψ (spin 1/2 mesons). The two coupling types which lead to $A_{kl}=1$ in the static limit are the scalar and vector coupling: $a=\beta$ and $a=1$. The scalar coupling is the one adopted by Wigner, Critchfield and Teller$^{4)}$ in their investigation of the single cluster problem, and the two-nucleon interaction for the same coupling case has been studied by Critchfield and Lamb,$^{5)}$ by Jauch,$^{9)}$ and more completely by Houriet (H$^{6)}$). Because of mathematical complications, these papers treat in detail only the weak and strong coupling cases ($\eta \ll R^2$, $\eta \gg R^2$). Substituting again, however, ηn_k for η, one sees easily that Houriet's strong coupling approximation is actually applicable if both ηn_1 and ηn_2 are $\gg R^2$, and then the mutual potential $V(r)$ becomes independent of n_1 and n_2,*** as required by (B). This includes single nucleons (n_1 and/or $n_2=1$) if $\eta \gg R^2$. In the vector coupling case, the situation is basically the same.

IV. Strong Coupling Approximation.

Turning to theories with $A_{kl} \neq 1$, we first choose as comparatively simple examples the spinor field with tensor and pseudovector coupling. Substituting

$$H_i=\eta \sum_k \boldsymbol{S}_k \cdot \bar{\psi}^*(\boldsymbol{x}_k)\beta\boldsymbol{\sigma}\bar{\psi}(\boldsymbol{x}_k), \tag{8}$$

$$H_i=\eta \sum_k \boldsymbol{S}_k \cdot \bar{\psi}^*(\boldsymbol{x}_k)\boldsymbol{\sigma}\bar{\psi}(\boldsymbol{x}_k). \tag{9}$$

Here, $\boldsymbol{\sigma}$ is the Dirac spin vector matrix of the ψ-field ($\sigma_z=i\alpha_x\alpha_y$, with eigenvalues $+1, -1, +1, -1$), and $\boldsymbol{S}_k$ is the total spin vector of the nucleonic cluster k, as is more fully explained in Appendix 1, in conjunction with a more detailed

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* For instance, the characteristic function $\varphi_2(\xi)$ introduced on p. 117 in W now becomes

$$\varphi_2(\xi)=\varphi_1^{(1)}(\xi)\varphi_1^{(2)}(\xi)-n_1n_2[\chi(r,\xi)]^2$$

where $\varphi_1^{(k)}$ is the function φ_1 with λ replaced by λn_k (cf. eq's (4)–(7) and (14) in W).

** To compare the notations: The quantities (lengths) η and R are called λ and $4\pi/A$ in W; also

$$J(r)=(\hbar\mu/32\pi^2)r^{-2}H_1^{(1)}(2i\mu r)$$

(μ=meson mass).

*** See eq's (25)–(27) in H$^{6)}$.

justification of the expressions (8) and (9), for the static limiting case. As to the nucleonic charge, we assume both protons and neutrons to be coupled with the same ψ-field, the coupling parameter being the same, in magnitude and sign.[†] The individual nucleon charges are, of course, constants of motion.

The problem (8), for $n_k=1$, has been solved by Blatt[7] in the strong coupling limit. The general procedure is to split the field operator $\psi(\boldsymbol{x})$ into parts having the $\boldsymbol{x}$-dependence of the various source functions $U(\boldsymbol{x}-\boldsymbol{x}_k)$ and another part $\psi'(\boldsymbol{x})$ orthogonal on every source function:

$$\psi(\boldsymbol{x})=\Gamma^{-1/2}\sum_k Q_k U(\boldsymbol{x}-\boldsymbol{x}_k)+\psi'(\boldsymbol{x}), \tag{10}$$

$$\int d^3x U(\boldsymbol{x}-\boldsymbol{x}_k)\psi'(\boldsymbol{x})=0, \quad k=1,\ 2,\ \cdots . \tag{11}$$

It is convenient to choose

$$\Gamma=\int d^3x U^2(\boldsymbol{x}). \tag{12}$$

Then, assuming that no overlapping of different source functions occurs (cf. (7)):

$$Q_k=\Gamma^{-1/2}\int d^3x U(\boldsymbol{x}-\boldsymbol{x}_k)\psi(\boldsymbol{x})=\Gamma^{-1/2}\overline{\psi}(\boldsymbol{x}_k), \tag{13}$$

resulting in the following anticommutation rules for the new field variables, or rather their spinor components ($\rho=1\cdots4$):

$$[Q^*_{k\rho},\ Q_{k\rho}]_+=1, \tag{14}$$

$$[\psi_\rho'^*(\boldsymbol{x}),\ \psi_\rho'(\boldsymbol{x}')]_+=\delta(\boldsymbol{x}-\boldsymbol{x}')-\Gamma^{-1}\sum_k U(\boldsymbol{x}-\boldsymbol{x}_k)U(\boldsymbol{x}'-\boldsymbol{x}_k), \tag{15}$$

while all other pairs of variables anticommute. The total Hamiltonian:

$$H=\int d^3x\psi^*(-i\boldsymbol{\alpha}\cdot\nabla+\beta\mu)\psi+H_i, \tag{16}$$

expressed in terms of the new variables becomes (taking spherical symmetry of $U(x)$ for granted)

$$H=\sum_k H_k+H'+\Omega, \tag{17}$$

$$H_k=\mu Q_k{}^*\beta Q_k+\eta\Gamma\boldsymbol{S}_k\cdot Q_k{}^*\varepsilon\boldsymbol{\sigma}Q_k \tag{18}$$

$$(\varepsilon=\beta \text{ for } (8),\ \varepsilon=1 \text{ for } (9)),$$

$$H'=\int d^3x\psi'^*(-i\boldsymbol{\alpha}\cdot\nabla+\beta\mu)\psi'. \tag{19}$$

Ω contains the terms bilinear in $Q_k^{(*)}$ and $\psi'^{(*)}$ which can be disregarded in the strong coupling limit ($\eta\gg\mu\Gamma^{-1}$) as has been shown by Blatt.

[†] The case of a different η for protons and neutrons (e.g. same magnitude but opposite signs) can be treated along the same lines. We ignore this case for the sake of simplifying the discussion.

Then, on account of the additivity of the H_k terms, the interaction between different clusters is solely determined by the H' part of the Hamiltonian, in conjunction with (11) and (15). As Blatt observed, the coupling type is quite irrelevant for this problem, the positions $\boldsymbol{x}_k$ being the only significant parameters. Indeed, the two-nucleon potential derived agrees with that obtained by Houriet for the scalar coupling case.* More important for the present discussion is the fact that also the quantity denoted by $\boldsymbol{S}_k$ has no bearing whatsoever on H' problem. Whether we have one or several nucleons at a point $\boldsymbol{x}_k$ is quite immaterial for the interaction energy. This is again the saturation effect (B), with the specification typical of the strong coupling limit that the critical value of the n_k's, above which the saturation sets in, is $\ll 1$.

V. The Single Cluster Problem.

For the discussion of the remaining terms in H (17), since they are additive in k, we can single out one term H_k and drop the suffix k:

$$H=\mu Q^*\beta Q+\omega\boldsymbol{S}\cdot Q^*\varepsilon\boldsymbol{\sigma}Q \qquad (\omega=\eta\Gamma). \tag{20}$$

We are interested in finding the stationary states, in particular the ground state, of the cluster, i.e. n nucleons at the same point. According to (14), the Q^* and Q components can be interpreted as creation and annihilation operators referring to particles in 4 states. Introducing the occupation numbers $Q_\rho^*Q_\rho=N_\rho$, with eigenvalues 0 and 1, and choosing the matrices β and σ_z diagonal, we may write

$$\left.\begin{aligned} Q^*\beta Q&=N_1+N_2-N_3-N_4,\\ Q^*\sigma_z Q&=N_1-N_2+N_3-N_4,\\ Q^*\beta\sigma_z Q&=N_1-N_2-N_3+N_4. \end{aligned}\right\} \tag{21}$$

Of course, the x and y components of $Q^*\varepsilon\boldsymbol{\sigma}Q$ are non-diagonal with respect to the occupation numbers N_ρ. The matrix $\boldsymbol{S}$, representing the total spin of the n nucleons, is conveniently reduced to its irreducible parts, with eigenvalues $S^2=s(s+1)$, where

$$s=\frac{n}{2},\ \frac{n}{2}-1,\ \cdots,\ \frac{1}{2}\ \text{or}\ 0, \tag{22}$$

and with appropriate multiplicities. Clearly, each s-value can be treated separately; this reduces the problem to a system $16(2s+1)$ linear equations the solution of which offers no difficulty. The resulting eigenvalues of H (20) and their multiplicities are collected in the Tables I and II, for the tensor and pseudovector coupling ($\varepsilon=\beta$ and 1) respectively. The order of the energy levels is primarily determined by the numerical coefficient of $\omega(=\eta\Gamma)$ because the condition for strong coupling is $\omega\gg\mu$. For $s=\frac{1}{2}$ and $\varepsilon=\beta$, our results agree with those of Blatt[7] (his $fN=\eta\Gamma/2$).

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* See footnote ***, page 588.

If the ψ-field carries a charge of density $\psi^*\psi$ (or rather $\psi^*\psi$ minus its vacuum expectation value), $Q^*Q=\sum_\rho N_\rho$ represents the charge of the mesons bound to the cluster, except for an additive constant. Applying hole theory ideas to the states $\rho=3$ and 4 (let $\eta\rightarrow 0$ adiabatically), we have to define the bound meson charge as

$$e=Q^*Q-2=N_1+N_2-(1-N_3)-(1-N_4). \tag{23}$$

e then has eigenvalues ranging from -2 to $+2$ (cf. the third column of the Tables).

Table I ($\varepsilon=\beta$).

Energy	Multiplicity	e
$2\omega(s(s+1))^{\frac{1}{2}}$	$2s+1$	0
$\omega(s+1)\pm\mu$	$\{2s$ $\{2s$	±1
$\omega s\pm\mu$	$\{2s+2$ $\{2s+2$	±1
$\pm2\mu$	$\{2s+1$ $\{2s+1$	0
0	$4(2s+1)$	$0,\pm2$
$-\omega s\pm\mu$	$\{2s+2$ $\{2s+2$	±1
$-\omega(s+1)\pm\mu$	$\{2s$ $\{2s$	±1
$-2\omega(s(s+1))^{\frac{1}{2}}$	$2s+1$	0

Table II ($\varepsilon=1$).

Energy	Multiplicity	e
$2\omega s$	$2s+3$	0
$\omega s\pm\mu$	$\{4(s+1)$ $\{4(s+1)$	±1
$\pm2\mu$	$\{2s+1$ $\{2s+1$	0
0	$3(2s+1)$	$0,\pm2$
$-\omega(s+1)\pm\mu$	$\{4s$ $\{4s$	±1
-2ω	$2s+1\}$ absent	0
$-2\omega(s+1)$	$2s-1\}$ for $s=0$	0

It is worth mentioning that the leading terms ($\sim\omega$) in Table II can be very easily checked by the following argument. With $\varepsilon=1$ and $\mu\rightarrow0$, the Hamiltonian (20) reduces to

$$H=2\omega\boldsymbol{S}\cdot\boldsymbol{L}, \quad \text{where} \quad \boldsymbol{L}=\frac{1}{2}Q^*\sigma Q. \tag{24}$$

$\boldsymbol{L}$ obeys the commutation rules of an angular momentum, and a reduced representation is readily found by observing that the eigenvalues of L_z (see (21)) are ±1 (single), $\pm\frac{1}{2}$ (4-fold), 0 (6-fold). Hence, $\boldsymbol{L}$ contains the following irreducible submatrices: $l=1$ (once), $l=\frac{1}{2}$ (4 times), $l=0$ (5 times) [$l(l+1)$ being, of course, the eigenvalues of L^2]. The introduction of the operator $\boldsymbol{J}=\boldsymbol{S}+\boldsymbol{L}$, with eigenvalues $J^2=j(j+1)$, leads immediately to the eigenvalues of H (24):

$$H(j,s,l)=\omega[j(j+1)-s(s+1)-l(l+1)],$$
$$|s-l|\leqq j\leqq s+l, \tag{25}$$

with multiplicities $2j+1$. The terms $+2\omega s$, -2ω, $-2\omega(s+1)$ in Table II form the triplet ($l=1$), and $+\omega s$, $-\omega(s+1)$ the four doublets ($l=\frac{1}{2}$). $\boldsymbol{J}$ is the total spin of the state, $\boldsymbol{L}$ representing the contribution of the bound mesons. (See, however, Appendix 2, last paragraph).

Before discussing these results, we wish to apply the same argument to another problem, involving, instead of the spins $\boldsymbol{s}$ and $\boldsymbol{\sigma}/2$, the isotopic spins. Consider, as usual, the proton and neutron as two states of the nucleon, belonging to $s_3=\pm\frac{1}{2}$ respectively, $\boldsymbol{s}$ now being interpreted as isotopic spin operator; $s_3+\frac{1}{2}$ is the charge of the nucleon, and the sum of n such charges, viz. $S_3+n/2$, represents the charge of a cluster.[††] Similarly, we assume the mesons to have charged and neutral states, described by a spinor field ψ with eight (instead of four) components. In the above formulas, e.g. (9), we reinterpret $\boldsymbol{\sigma}/2$ to mean the isotopic spin operator of the ψ-field, so that $\psi^*(\sigma_3+1)\psi/2$ (minus its vacuum value) is the mesonic charge density. The strong coupling approximation leads again to (17), (18), (19), and (20); but instead of (21), one has, for instance;

$$\begin{aligned} Q^*\sigma_3 Q = N_1+N_2+N_3+N_4 \\ -N_5-N_6-N_7-N_8, \end{aligned} \tag{26}$$

with eigenvalues ranging from $+4$ to -4. The quantity

$$\frac{1}{2}Q^*(\sigma_3+1)Q-2=N_1+N_2-(1-N_3)-(1-N_4) \tag{27}$$

signifies the total charge of the mesons bound to the nucleon cluster.

Now, let us adopt the Hamiltonian (9), or (20) with $\varepsilon=1$, reinterpreted as just indicated. H is invariant under rotations in the isotopic spin space, thus yielding a "charge-symmetric" theory.[10] Because of the absence of other spin-operators in H (corresponding to $a=1$ in (4)), the coupling adopted is a vector coupling (static limit). With $\mu\to 0$, H again reduces to (24), and the solution is obtained as above, with the only difference that l now assumes the values 0, $\frac{1}{2}$, 1, 3/2, 2, as is evident from (26). Except for this extension, the eigenvalues are determined by (25). The total charge of a state, resulting from nucleonic and mesonic charges (cf. (27)), is

$$\begin{aligned} e&=\left(S_3+\frac{n}{2}\right)+\left(L_3+\frac{1}{2}Q^*Q-2\right) \\ &=J_3+\frac{n}{2}+\frac{1}{2}(Q^*Q-4). \end{aligned} \tag{28}$$

Here, $Q^*Q=N_1+\cdots+N_8$ is a constant of motion. For later purposes we note that the maximum l-value, vis. $l=2$, can only be attained if $N_1+N_5=N_2+N_6=N_3+N_7=N_4+N_8=1$, which implies $Q^*Q=4$. Hence, according to (28):

$$e=J_3+n/2, \quad \text{for} \quad l=2, \tag{29}$$

where $J_3=-j,\ -j+1,\cdots\cdots,\ j-1,\ j$.

VI. The Self-Potential.

Let E_n be the lowest energy eigenvalue of the cluster containing n nucleons. Then,

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[††] The argumentation of Appendix 1 can obviously be applied to this case, *mutatis mutandis*.

$$V_n = E_n - nE_1 \tag{30}$$

is the self-potential of the cluster, i.e., the energy required for assembling the cluster. If $V_n > 0$, the system is certainly unstable; if $V_n < 0$, $|V_n|$ may be called the binding energy of the cluster.

It should be noted that also the H' part of the Hamiltonian (17) contributes to E_n. Since the H' problem does not involve n, its contribution to E_n is independent of n (i.e. $E_n' = E_1'$) and yields a potential:

$$V_n' = -(n-1)E_1'. \tag{31}$$

Remembering that the H' problem is the same for all coupling type considered, including the scalar coupling,[4] the value of E_1' is known; since it is positive, the potential (31) is attractive. But E_1' is by definition independent of the coupling constant and, therefore, insignificant compared with $\omega = \eta\Gamma$, in the strong coupling limit.

In scrutinizing the terms $\sim\omega$, we start with the tensor coupling theory, Table I. Here, the sign of ω does not affect the level scheme, and we can choose $\omega > 0\,(\eta > 0)$. The lowest energy value in the Table is $-2\omega(s(s+1))^{\frac{1}{2}}$, and inserting for s its maximum value $n/2$ (cf. (22)), we obtain for the ground state:

$$\left.\begin{aligned} E_n &= -\omega(n(n+2))^{1/2} + E_n', \\ V_n &= \omega n^{1/2}[(3n)^{1/2} - (n+2)^{1/2}] + V_n'. \end{aligned}\right\} \tag{32}$$

V_n is positive for $n > 1$, thus prohibiting the formation of the cluster.

In the case of pseudovector coupling, Table II, the situation is quite different. If $\eta < 0$, $\omega < 0$, the lowest energy listed in the Table is $-2|\omega|s$; consequently:

$$E_n = -|\omega|n + E_n', \quad V_n = V_n'. \tag{33}$$

Due to the cancellation of the ω-terms, the self-potential reduces to (31) corresponding to attractive forces having the saturation property (A) (see Section II). A less desirable feature is that the ground state of the cluster has the spin $j = s + l = \frac{1}{2}n + 1$. Only for this state, in which all individual spins (of nucleons and bound mesons) are lined up to yield the maximum total spin, are the forces attractive.

A similar difficulty is encountered in the charge-symmetrical theory, as specified in Section V, for $\eta < 0$. Again one finds $V_n = V_n'$ for the ground state, but this state, which is the only stable one, has the total isotopic spin $j = s + l = \frac{1}{2}n + 2$, meaning that the total charge e (29) can assume values ranging from -2 to $n+2$. In order to attain these extreme values the n nucleons have to be either all in the neutron, or all in the proton state, while the bound meson contribute -2 or $+2$ charge units respectively.

Much more satisfactory are the results if we assume $\eta > 0$. Both for the pseudovector and for the charge-symmetrical theory, described in Section V, the ground state energies are then given by (25),[†††] with $l=1$ and 2, respectively, $s=n/2$, and $j=|s-l|$:

$$\left.\begin{aligned} E_n-E_n'&=H\left(\left|\frac{1}{2}n-l\right|, \frac{1}{2}n, l\right) \\ &=\begin{cases} -\omega(l+1)n, & \text{if } n\leqq 2l, \\ -\omega l(n+2), & \text{if } n\geqq 2l. \end{cases} \end{aligned}\right\} \tag{34}$$

(In comparing with Table II, for $l=1$ and $s=n/2$, one should notice that the lowest state listed there has the multiplicity $2s-1$ and is non-existent for $s=\frac{1}{2}$ or $n<2$.) Then, according to (30):

$$V_n-V_n'=\begin{cases} 0 & , \text{ if } n\leqq 2l, \\ \omega(n-2l), & \text{if } n\geqq 2l. \end{cases} \tag{35}$$

In words: V_n is strongly repulsive for $n>2l$ and reduces to V_n' (31) for $n\leqq 2l$. In the charge-symmetrical theory, where $l=2$, this means that the biggest cluster which can form at one point contains four nucleons, and this cluster is an "α-particle" because $j=|s-l|=0$, $J_3=0$, $\epsilon=n/2=2$ (cf. (29)). Table III lists in addition the smaller clusters which are stable. For $n=3$ they correspond to the known nuclei He^3 and H^3; for $n=2$, the di-proton and the di-neutron appear with the same binding energy as the "deuteron" (which is a necessary consequence of the charge-symmetry and spin-independence of the Hamiltonian). More disturbing is that, for $n=1$, the proton and neutron appear associated with "isobaric states" of charges 2 and -1. For all these states, the binding energy is just $(n-1)E_1'$, according to (31). Needless to say that our model is too simple and crude to reproduce the finer features of nuclear binding. But the fact that it allows very few stable states unless the nucleus spreads over a larger volume ($>R^3$) may be significant. For a nucleus of higher atomic weight, the most stable structure will resemble an "alpha-particle model". It should be observed that the Pauli exclusion principle, as applied to the nucleons, has played no decisive part in the derivation of these results.

In the (charge-independent) pseudovector-coupling theory (9) ($\eta>0$), where (35) holds with $l=1$, at most two nucleons can form a stable cluster, with spin $j=|s-l|=0$. This case was used as a test case to study the further question how the selfpotential V_2 changes if the two nucleons, originally located at the same point, are separated to a small distance r. Because of the overlapping of the two source functions, the formulae of Section IV (starting from eq. (13)) are then no longer applicable, and a different method had to be applied which

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[†††] The first term in (20) which was omitted in (25) gives no contribution because $Q^*\beta Q=0$ for the states in question.

Table III.

n	s	l	j	J_3	e
1	1/2	2	3/2	±3/2, ±1/2	2, 1, 0, −1
2	1	2	1	±1, 0	2, 1, 0
3	3/2	2	1/2	±1/2	2, 1
4	2	2	0	0	2

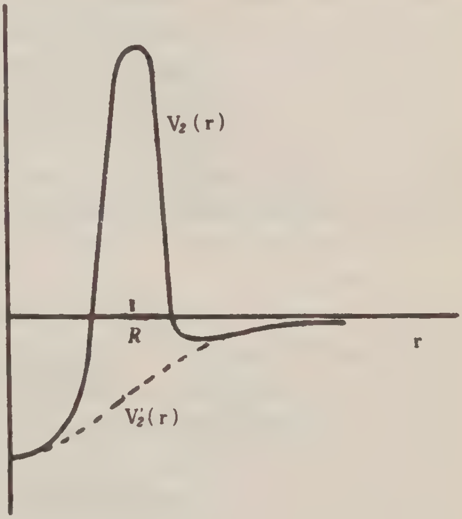

Fig. 1.

is outlined in Appendix 2. The result is qualitatively described by Figure 1 (the full-drawn curve represents the total potential, whereas the dotted line indicates the contribution of H'). The high potential barrier (at $r \sim R$) will render the once formed cluster even more stable. (The vibrational zero-point energy of the two-nucleon system inside the well is much smaller than the barrier height.) In the outside region, where the overlap becomes insignificant, V_2 reduces to V_2' (<0). This part of the potential, as we know from Section IV, is not altered if one or both nucleons are replaced by clusters; in other words, the attraction between clusters at larger distances is not impaired by the presence of the potential barrier.

VII. Summarizing Remarks.

What we may learn from our sample theories is primarily that the nuclear forces deducible from pair theories are of a much richer variety than could be anticipated from previous work. Nuclear structure models ranging from single-nucleon to α-particle models are suggested. Since either of these models has proved fruitful in explaining certain nuclear phenomena one may wonder whether a theory can be set up that combines the main features of both models. This, however, seems hard to achieve. The very fact that a complete α-unit, of the type discussed in Section V, will act as a strongly repulsive center on every outside nucleon spoils the single particle picture. Even an unsaturated cluster, if it is surrounded by a potential barrier like that described in Figure 1, would cut short the mean free path of an approaching nucleon.

Nevertheless, considering that the most conspicuous properties of nuclei are those of a liquid drop model, the most interesting phenomenon resulting from the above theories seems to be that

(D) the population of a small cell cannot be increased beyond a certain maximum occupation number n_s, except at high cost in energy.

The value of n_s depends, of course, on the interaction type; indeed, we found

$n_s=1$ for the tensor coupling theory (18), $n_s=2$ for the pseudovector coupling (9) with $\eta>0$, and $n_s=4$ (α-particle) for the charge-symmetric vector coupling theory with $\eta>0$. According to these and similar theories, a big nucleus will behave like a liquid drop which cannot be compressed beyond a certain maximum density. It seems very likely that all pair theories, except those with $A_{\lambda\lambda}=1$ (cf. Section III), lead to the saturation effect (D). Further effort is needed to explore other theories, in particular the pseudoscalar coupling case, but unfortunately the mathematical complications are considerable.

Appendix 1.

It may be desirable to justify eq's (8), (9), as expressing the tensor and pseudovector interactions in the static limiting case.

In a non-relativistic approximation (Pauli), the nucleonic field operator ϕ may be written:

$$\phi=\sum_{m,\lambda}a_{m\lambda}\chi_\lambda u_m(\boldsymbol{x}) \quad (\lambda=1\cdots4),$$

$$[a_{m\lambda}^*,\ a_{m'\lambda'}]_+=\delta_{mm'}\delta_{\lambda\lambda'}, \quad \text{etc.},$$

where χ_λ describes the spin and charge state and u_m the translational state of a nucleon. If all nucleons are to form a " small cluster ", choose for the u_m the Schroedinger eigenfunctions of a potential well, of lowest possible energies; the spacing of the energy levels is of the order $M^{-1}d^{-2}$ (M=nucleon mass, d=width of well) and tends to zero in the static limit ($M\rightarrow\infty$). [Note that the total energy interval required, even for $d\lesssim R\sim\mu^{-1}$, can be made $<\mu(\sim M/10)$ unless the number of nucleons is excessively large. This is sufficient for our purposes.]

Now, returning to the general expression for H_1 (1), with $A=\boldsymbol{s}=\boldsymbol{\sigma}/2$ and $a=\boldsymbol{\sigma}$ or $\beta\boldsymbol{\sigma}$, define:

$$\boldsymbol{S}=\int d^3x\phi^*\boldsymbol{s}\phi=\sum_m \boldsymbol{s}_m,$$

$$\boldsymbol{s}_m=\sum_{\lambda,\lambda'}a_{m\lambda}^*(\lambda|\boldsymbol{s}|\lambda')a_{m\lambda'}.$$

The individual $\boldsymbol{s}_m$-operator as well as $\boldsymbol{S}$ obey the commutation rules of angular momenta. If the translational state m is occupied by a single nucleon, $\boldsymbol{s}$ is the spin operator of this nucleon, and if several nucleons occupy the same state m, $\boldsymbol{s}_m$ is their total spin. This is easily seen by introducing the occupation number $a_{m\lambda}^*a_{m\lambda}=N_{m\lambda}$ (=0 or 1) of the substates λ (=1⋯4) and considering the $\boldsymbol{s}_m$-components as matrices with respect to these numbers [e.g. $s_{mz}=1/2\ (N_{m1}-N_{m2}+N_{m3}-N_{m4})$, with eigenvalues 0, $\pm1/2$, ±1]. Thus $\boldsymbol{S}$ means the total spin of the cluster. S^2 has the eigenvalues $s(s+1)$, with:

$$s=n/2,\ n/2-1,\cdots\cdots\cdots\cdots,\ 1/2 \text{ or } 0$$

(n=total population of the cluster). The maximum s-value can of course be realized only if each translational state contains one proton and one neutron at most.

Finally, we exploit the above-mentioned fact that, in the static limit $M \to \infty$, the size of the cluster can be made arbitrarily small ($d \ll R$) which allows us to express the spin density as $\boldsymbol{S}$ times a delta function. Generalizing to the case of several clusters, with total spins $\boldsymbol{S}_k$, we may write

$$\phi^* \boldsymbol{s} \phi = \sum_k \boldsymbol{S}_k \delta(\boldsymbol{x} - \boldsymbol{x}_k).$$

Inserting this into (1), and remembering (3), we obtain (8) or (9).

Appendix 2.

We want to study the two-nucleon problem with the interaction law (9):

$$H_i = \eta \sum_{k=1}^{2} \boldsymbol{s}_k \cdot \bar{\psi}^*(\boldsymbol{x}_k) \boldsymbol{\sigma} \bar{\psi}(\boldsymbol{x}_k),$$

where $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ denote the individual nucleon spins. In order to find the leading terms ($\sim \omega = \eta \Gamma$) in the potential energy, we need only calculate the eigenvalues of H_i, since the other (free field) terms in H (16) contribute nothing of comparable order, if $\omega \gg \mu$. Note that the overlap integral

$$\Delta = \int d^3x U(\boldsymbol{x} - \boldsymbol{x}_1) U(\boldsymbol{x} - \boldsymbol{x}_2)$$

need not vanish. We define

$$\Gamma_+ = \frac{1}{2}(\Gamma + \Delta), \quad \Gamma_- = \frac{1}{2}(\Gamma - \Delta)$$

(both > 0, cf. (12)), and introduce the following two sets of variables:

$$q_+ = \frac{1}{2} \Gamma_+^{-1/2} [\bar{\psi}(\boldsymbol{x}_1) + \bar{\psi}(\boldsymbol{x}_2)]$$

$$q_- = \frac{1}{2} \Gamma_-^{-1/2} [\bar{\psi}(\boldsymbol{x}_1) - \bar{\psi}(\boldsymbol{x}_2)]$$

$$[q_{+\rho}^*, q_{+\rho}]_+ = [q_{-\rho}^*, q_{-\rho}]_+ = 1$$

(all other pairs anticommute):

$$q_1 = 2^{-1/2}(q_+ + q_-), \quad q_2 = 2^{-1/2}(q_+ - q_-)$$

$$[q_{1\rho}^*, q_{1\rho}]_+ = [q_{2\rho}^*, q_{2\rho}]_+ = 1$$

(all other pairs anticommute). Expressed in terms of either of these variables:

$$\begin{aligned} H_i = \eta \{ & (\boldsymbol{s}_1 + \boldsymbol{s}_2) \cdot (\Gamma_+ q_+^* \boldsymbol{\sigma} q_+ + \Gamma_- q_-^* \boldsymbol{\sigma} q_-) \\ & + (\boldsymbol{s}_1 - \boldsymbol{s}_2) \cdot (\Gamma_+ \Gamma_-)^{1/2} (q_+^* \boldsymbol{\sigma} q_- + q_-^* \boldsymbol{\sigma} q_+) \}; \end{aligned}$$

$$\begin{aligned} H_i = \frac{\eta}{2} \{ & (\boldsymbol{s}_1 + \boldsymbol{s}_2) \cdot (\Gamma [q_1^* \boldsymbol{\sigma} q_1 + q_2^* \boldsymbol{\sigma} q_2] + \Delta [q_1^* \boldsymbol{\sigma} q_2 + q_2^* \boldsymbol{\sigma} q_1]) \\ & + (\boldsymbol{s}_1 - \boldsymbol{s}_2) \cdot (\Gamma^2 - \Delta^2)^{1/2} (q_1^* \boldsymbol{\sigma} q_1 - q_2^* \boldsymbol{\sigma} q_2) \}. \end{aligned}$$

Besides the exchange operator (interchanging the suffixes 1 and 2), the quantities $\sum_k q_k^*(1\pm\beta)q_k$, I_z, I^2 [where $\boldsymbol{I}=\sum_k(\boldsymbol{s}_k+\frac{1}{2}q_k^*\boldsymbol{\sigma}q_k)$] are constants of motion, and their eigenvalues are adiabatic invariants (viz. under adiabatic changes of Δ).

Now, consider two limiting cases:

(a) $r=|\boldsymbol{x}_1-\boldsymbol{x}_2|\ll R$, $\Delta\approx\Gamma$, $\Gamma_-\ll\Gamma_+$. In the limit $r\to 0$, $\Gamma_-\to 0$:

$$H_i\to H_i^0=\eta\Gamma(\boldsymbol{s}_1+\boldsymbol{s}_2)\cdot q_+^*\boldsymbol{\sigma}q_+\quad(q_+\to Q).$$

This is the Hamiltonian (20), with $\mu=0$, $\varepsilon=1$, $n=2$, whose solutions are known from Section V. Treating the remainder $(H_i-H_i^0)$ as a small perturbation, we obtain (by a second order approximation) the potential energy up to terms linear in Γ_-.

(b) $r>R$, $|\Delta|\ll\Gamma$. In the limit $\Delta\to 0$:

$$H_i\to\eta\{\boldsymbol{s}_1\cdot q_1^*\boldsymbol{\sigma}q_1+\boldsymbol{s}_2\cdot q_2^*\boldsymbol{\sigma}q_2\}$$

which is the problem of two isolated nucleons. Again, a perturbational calculation (second order) gives the potential up to terms of the second order in Δ.

Applying these formulas to the lowest states of the unperturbed systems (for $\eta>0$), the second order approximation was actually carried through for case (a), whereas the limiting case (b) furnished the proper values of the adiabatic invariants. For those lowest states of the cluster which can be formed out of two separate ground state nucleons by joining them together adiabatically, one obtains as the leading term ($\sim\omega$) of the potential $V_2(r)=E_2(r)-2E_1$, in case (a):

$$V_2(r)=\frac{19}{6}\eta\Gamma_-+\cdots\sim+\omega(r/R)^2+\cdots\quad(r\ll R).$$

For case (b), we know at least the order of magnitude of the leading term:

$$V_2(r)\sim\omega(\Delta/\Gamma)^2+\cdots\quad(r>R).$$

With regard to the smaller terms (indicated by the dots), we need only remember that they reduce to $V_2'(r)$ (see Section VI) in both limiting cases ($\Gamma_-=0$ and $\Delta=0$). To complete the qualitative picture of the potential barrier, we ought to know, for case (b) ($r>R$), the sign of $V_2(r)-V_2'(r)[\sim\pm\omega(\Delta/\Gamma)^2]$. In Figure 1 we have assumed the positive sign as more likely to be the correct one.

One more remark about case (a) is necessary. In the limit $\Gamma_-\to 0$, the variables q_1 cease to contribute to the energy. This causes an additional degeneracy of the energy levels listed in Table II ($s=1$). In particular, the ground state becomes four-fold (instead of single). Of the total spin

$$\boldsymbol{I}=\boldsymbol{s}_1+\boldsymbol{s}_2+\frac{1}{2}q_+^*\boldsymbol{\sigma}q_++\frac{1}{2}q_-^*\boldsymbol{\sigma}q_-,$$

only the first three terms ($\boldsymbol{S}+\boldsymbol{L}=\boldsymbol{J}$) have been considered in Section V, yielding $j=0$ for the ground state, whereas the fourth term, arising from the q_- degrees of freedom, changes the eigenvalues of I_z to 0, 0, +1, −1, and causes the ground state to split into a singlet and a triplet state as r increases. In the terms $\sim\Gamma_-$ or $\sim r^2$ (see the above formula for $V_2(r)$, $r \ll R$) the singlet and triplet energies still coincide, but in the terms $\sim\Gamma_-^2$ or $\sim r^4$ the splitting actually occurs. In this last argument we have completely disregarded the free field Hamiltonian which would introduce further complications mainly by coupling the spins with the $\boldsymbol{r}$ direction, in the manner of tensor forces (term proportional to $(\boldsymbol{I}\cdot\boldsymbol{r})^2$ in $V_2(r)$).

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

The Reversal of Time and the Quantization of the Longitudinal Field in Quantum Electrodynamics

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(Received July 22, 1950)

Introduction.

It is well known that the equations of classical electrodynamics are invariant against the following improper Lorentz-transformation:

$$x_4' = -x_4, \qquad x_r' = x_r; \qquad r=1,2,3. \tag{1a}$$

$$\begin{aligned} A_4'(\boldsymbol{x}, t) &= -A_4(\boldsymbol{x}, -t), \\ A_r'(\boldsymbol{x}, t) &= A_r(\boldsymbol{x}, -t), \end{aligned} \tag{1b}$$

$$\begin{aligned} j_4'(\boldsymbol{x}, t) &= -j_4(\boldsymbol{x}, -t), \\ j_r'(\boldsymbol{x}, t) &= j_r(\boldsymbol{x}, -t). \end{aligned} \tag{1c}$$

The physical interpretation of (1) is that classical electrodynamics is invariant against a reversal of time if, at the same time, the sign of all charges is reversed. That j_r does not change sign, is due to the fact that it contains a factor e as well as the velocity of the particles. The Lorentz-condition $\partial A_\mu/\partial x_\mu=0$ remains invariant too and so does the conservation law $\partial j_\mu/\partial x_\mu=0$. It is to be expected that in quantum-electrodynamics a similar transformation is possible. The mathematical problem is to find a transformation of the state vector

$$\psi' = S\psi \tag{2}$$

such that; (i) the equation of motion for ψ' is the same as for ψ, regarding all operators as fixed and (ii) the expectation values of the field quantities transform like (1b, c).

For simplicity we regard j_μ as a c-number field and then, in the transformed equation of motion j_μ' appears instead of j_μ. The transformation in question takes a straightforward form in the "abbreviated theory" in which the longitudinal field is eliminated and replaced by the static potential

$$V(t) = \iint \frac{\rho(\boldsymbol{x}, t)\rho(\boldsymbol{x}', t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x\, d^3x', \qquad j_4 = i\rho.$$

Then $V'(t) = V(-t)$. The state vector is $\psi(\cdots\cdots N_l^\lambda\cdots\cdots; t)$, where N_l^λ is the number of photons with momentum l and polarization $\lambda=1,2$. The operator of the transverse field $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\sum_{k,\lambda}\boldsymbol{e}_\lambda(k)\left\{a_k^{\lambda *}e^{-i(\boldsymbol{k}\boldsymbol{x})+i\nu t}+a_k^\lambda e^{i(\boldsymbol{k}\boldsymbol{x})-i\nu t}\right\},$$

where a_k^λ, $a_k^{\lambda *}$ are the absorption and emission operators, and $\boldsymbol{e}$ the polarization unit vector. We denote the hermitian conjugate of any operator by *. The wave equation is

$$i\frac{\partial\psi(t)}{\partial t}=V(t)\psi(t)-\{\textstyle\int\boldsymbol{j}(\boldsymbol{x},t)\boldsymbol{A}(\boldsymbol{x},t)d^3x\}\psi(t). \tag{3}$$

In order to find ψ' we go over to the complex conjugate equation of (3) and at the same time reverse the sign of t. Then $\psi^*(\cdots N_\iota^\lambda\cdots\cdots;-t)$ satisfies the same equation (3) with V replaced V', $\boldsymbol{j}$ replaced by $\boldsymbol{j}'$ (1b), and $\boldsymbol{A}$ replaced by

$$\sum_{k,\lambda}\boldsymbol{e}_\lambda(k)\left\{a_k^{\lambda *}\mathrm{e}^{i(\boldsymbol{k}\boldsymbol{x})+i\nu t}+a_k^\lambda e^{-i(\boldsymbol{k}\boldsymbol{x})-i\nu t}\right\}. \tag{4}$$

For convenience we have chosen the matrix-elements of a_k and a_k^* to be real. Next we introduce new occupation numbers $\overline{N}$

$$\overline{N}_\iota^\lambda=N_{-\iota}^\lambda, \tag{5a}$$

where $N_{-\iota}$ is the number of photons with momentum $-\boldsymbol{\iota}$ and define a new state vector ψ'

$$\psi'(\cdots\overline{N}_\iota^\lambda\cdots\cdots;-t)=\psi^*(\cdots N_\iota^\lambda\cdots\cdots;t). \tag{5b}$$

The wave equation satisfied by ψ' is obtained, if we also replace in (4) a_k by a_{-k}, a_k^* by a_{-k}^*, (but leave the exponentials unchanged). Changing now the summation variable $\boldsymbol{k}$ into $-\boldsymbol{k}$, it is seen that the operator $\boldsymbol{A}$ is restored completely (as it should) and ψ' satisfies

$$i\frac{\partial\psi'}{\partial t}=V'\psi'-\{\textstyle\int\boldsymbol{j}'\boldsymbol{A}\,d^3x\}\,\psi'. \tag{6}$$

Thus (5b) is the desired transformation.

The reversal of the sign of all photon-momenta corresponds to the fact that classically all velocities are reversed. In the same way it is also easily verified that the expectation values $\overline{\boldsymbol{A}}=(\psi^*\boldsymbol{A}\psi)$, $\overline{\boldsymbol{A}}'=(\psi'^*\boldsymbol{A}\psi')$ are related by the classical transformation (1a):

$$\overline{\boldsymbol{A}}'(\boldsymbol{x},t)=\overline{\boldsymbol{A}}(\boldsymbol{x},-t). \tag{7}$$

§ 1. Quantization by usual methods.

We consider now the same problem in the theory where the longitudinal field is not eliminated and consequently 4 types of photons exist. We denote the corresponding numbers by N_ι^μ, $\mu=1\cdots4$. The wave equation is now

$$i\frac{\partial\psi(t)}{\partial t} = -\{\int j_\mu \cdot A_\mu d^3x\}\psi(t) \tag{8}$$

with the auxiliary condition (X, $\bar{X}$ are 4-vectors)

$$\Omega(\bar{X}, t)\psi(t) \equiv \left\{\frac{\partial A_\mu(\bar{X})}{\partial X_\mu} + \int_{X_4=it} D(\bar{X}-X)\rho(X)d^3X\right\}\psi(t) = 0 \tag{9}$$

to be satisfied for all $\bar{X}$. We regard again j_μ as a c-number field. One should think at first that the same transformation (5b) with $\bar{N}^\mu_\iota = N^\mu_{-\iota}$ for all 4 types of photons should leave (8) invariant. However, this is not the case, and a special treatment is required for the fourth component and N^4_ι. In this theory $A_4 = iA_0$, and A_0, being classically real, is a hermitian operator.

$$A_\mu = \sum_k \left\{a_k^{\mu*} e^{-i(\boldsymbol{k}\boldsymbol{x})+i\nu t} + a_k^\mu e^{i(\boldsymbol{k}\boldsymbol{x})-i\nu t}\right\}$$

with (10)

$$a_k^4 = i\,a_k^0, \qquad a_k^{4*} = i\,a_k^{0*},$$

where now, owing to the change of sign in the commutation relations for A_0, a_k^0 is the emission, a_k^{0*} the absorption operator, whereas a_k^r, a_k^{r*} have the usual meaning. Proceeding now as above and denoting for a moment

$$\psi''(\cdots\bar{N}^\mu_\iota\cdots;\ -t) = \psi^*(\cdots N^\mu_\iota\cdots;\ t),$$

we obtain

$$i\frac{\partial\psi''}{\partial t} = -\{\int (j'_r A_r + j'_0 A_0)\, d^3x\}\psi''. \tag{11}$$

The wrong sign of the term $j'_0 A_0$ arises from the fact that $j'_0(t) = -j_0(-t)$, according to (1b). It follows that ψ'' is not yet the correct transformed state vector. To obtain the correct transformation of ψ we need a unitary operator η with the following properties:

$$\eta^* = \eta^{-1}, \qquad \eta A_r = A_r\eta, \qquad \eta A_4 = -A_4\eta, \tag{12a}$$

$$\eta^2 = 1. \tag{1Pb}$$

(12b) is necessitated by the group property of the time reversal. It follows then that η is also hermitian $\eta^* = \eta$ which will be important in § 2. In our representation where N^4_ι is diagonal, η is also diagonal, $\eta = (-1)^{N^4}$, where N^4 is the total number of scalar photons. This operator is very similar to that occurring in the indefinite metric of Dirac and Pauli (the parts operating on N^4 are identical) although η occurs here in a very different connection. In § 2 we shall see that time reversal takes a simpler form if the scalar field is quantized by indefinite metric from the start. We define now

$$\psi'(\cdots\bar{N}^\mu_\iota\cdots;\ -t) = \eta\psi'' = \eta\psi^*(\cdots N^\mu_\iota\cdots;\ t). \tag{13}$$

It is immediately seen that the wave equation is now correct (multiply (11) by η from the left and use (12a))

$$i\frac{\partial\psi'}{\partial t}=-\{\int j'_\mu A_\mu d^3x\}\psi'. \tag{14}$$

The auxiliary condition (9) is also covariant under the transformation in question. D changes sign when $X_4\to -X_4$, $\overline{X}_4\to -\overline{X}_4$, and the same is true for ρ. So the second term becomes $D\rho'$. $\partial A_r/\partial X_r\cdot\psi$ goes over in $\partial A_r/\partial X_r\psi''$, if we first take the complex conjugate, reverse t and replace N_ι by $\overline{N}_\iota=N_{-\iota}$. $\partial A_0/\partial X_0\psi$ goes, however, over into $-\partial A_0/\partial X_0\psi''$. Multiplying by η from the left, η commutes with all terms except A_0, and restores the correct sign in the first term so that

$$\left\{\frac{\partial A_\mu}{\partial \overline{X}_\mu}+\int_{X_4=it} D\rho' d^3X\right\}\psi'=0. \tag{14a}$$

It is again easily verified that the expectation values of *all four* A_μ have now the same transformation properties as the classical quantities (1a).

§ 2. Quantization by the indefinite metric.

A different method of quantization for the fourth component A_4 has been proposed by Gupta[1] and one of us (B)[2] in order to remove the well known inconsistencies which prevail if the usual method of § 1 is used. It is well known that then a normalizable state vector does not exist. Moreover, the condition for the photon vacuum cannot be stated consistently in the form

$$A_\mu^+\psi=0, \tag{15}$$

because A_4^+ is then the emission operator and besides (15) is incompatible with the auxiliary condition (9). It is true that in the applications (15) need not be used explicitly but it is somewhat disconcerting to find that the results (which agree with those of the theory where the longitudinal field is eliminated) are precisely the same, as if (15) were actually used. These inconsistencies can be removed if the scalar field is quantized by means of the indefinite metric. In this theory the operator η, with precisely the properties stated already in (12), is introduced in the quantization of the field itself. This means that the scalar part of the field is quantized by indefinite metric, whereas for the space parts ordinary metric is applied. In contrast to § 1, A_4 is taken as hermitian and a^4, a^{4*} are the absorption and emission operators, as for the space components. A_0 defined as $A_0=\eta A_4$ is antihermitian. One then introduces the adjoint state vector

$$\psi^\dagger=\psi^*\eta \tag{16}$$

and the norm is $(\psi^\dagger\psi)=(\psi^*\eta\psi)$. The expectation value of any operator Q is now

$$\bar{Q}=(\psi^\dagger Q\psi). \tag{17}$$

It has been shown in ref.[2)] that then $\bar{A}_0$ is real (although A_0 is antihermitian) and $\bar{A}_4$ is purely imaginary. The auxiliary condition is modified, (taking the possible frequency part of Ω only):

$$\Omega^+\psi\equiv\left\{\frac{\partial A_\mu^+}{\partial X_\mu}+\int D^+\rho d^3X\right\}_{X_4=ct}\psi=0. \tag{18}$$

The possible states ψ, satisfying this condition have been studied in detail in references 1)2), for the vacuum $\rho=0$. They have a *non-negative* norm (which is a result of the auxiliary condition): $(\psi^\dagger\psi)\geq 0$ and the case $(\psi^\dagger\psi)=0$ can be excluded. So it is possible to normalize

$$(\psi^\dagger\psi)=1. \tag{19}$$

(15) is now a consistent vacuum condition and is compatible with (18) (for $\rho=0$). From (18) it follows also that

$$\psi^\dagger\Omega^-=0 \qquad \text{and therefore} \qquad (\psi^\dagger\Omega\psi)=0 \tag{20}$$

i.e. the expectation value $\bar{\Omega}$ vanishes which is sufficient to safequard the transition to classical theory. Proper Lorentz-transformations have also been discussed already in ref. 2) and it was shown that the theory is invariant. Time reversal takes here a very simple form. The wave equation is again (8) and for A_μ we can use the expansion (10) where now all a_k^μ are the usual absorption and all $a_k^{\mu *}$ emission operators. Therefore in contrast to § 1 also a_k^4 and a_k^{4*} can be chosen to have real matrix-elements. We proceed as before and define

$$\psi'(\cdots\bar{N}_k^\mu\cdots;\ -t)=\psi^*(\cdots N_k^\mu\cdots;\ t) \tag{21}$$

without the η operator. The space components j_rA_r go over, as before, into $j_r'A_r$. But this is now also the case for j_4A_4 without change of sign: Since A_4 is in no way distinguished from A_r, and has the same reality properties, A_4 is reproduced under the transformations involved in (12). j_4 is purely imaginary and changes sign in the complex conjugate equation, but the reversal of t changes the sign (according to (1b)). So ψ' defined now by (21) satisfies the wave equation (8). Thus the final transformation is now (21), which is what one expects intuitively, the η operator being incorporated in the method of quantization. The auxiliary condition (18) also remains invariant: All 4 A_μ^+ are reproduced (this holds for A_μ^+ and A_μ^- separately) and so is $x_4=it$ when we go over to the complex conjugate equation and reverse t. Likewise $D^+(t)=\frac{1}{2}(D+iD_1)$ just changes sign if we change the sign of t and that of i (because $D_1(t)=D_1(-t)$). So we have again

$$\Omega^{+\prime}\psi'=0, \tag{22}$$

where $\Omega^{+\prime}$ differs from Ω^+ in that ρ is replaced by ρ'. It is again easy to see that the expectation values of A_μ defined by $(\psi^\dagger A_\mu \psi)$ are transformed like the classical quantities, and that they have the same reality properties has already been mentioned. The present method seems to us slightly more preferable to that of § 1 (quite apart from the more serious difficulties involved in the method of § 1), because the transformation of the state vector for impróper Lorentz-transformations can be stated in a symmetrical form for all types of photons and is what one expects on general grounds and since the operator η cannot be avoided anyhow it might as well be used at a more fundamental point.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On Some Developments in Non-local Field Theory

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(Received July 26, 1950)

I. Introduction.

The idea of non-local fields has been suggested some years ago by H. Yukawa in a series of papers in Progress of Theoretical Physics and in Physical Review(1). It is essentially an attempt to introduce consistently into field theory a fundamental length which can be expected to be in some sense a radius of the elementary particles. A non-local field $U^{(1)}$ is a function both of the four space-time operators x^{μ} ($x^1=x_1=x$, etc...$x^4=-x_4=ct$) and of four space-time displacement operators p_{μ}, which satisfy the usual commutation relations

$$[x^{\mu},\ p_{\nu}]=\delta^{\mu}_{\nu}, \tag{1}$$

where

$$i\,[A, B]=AB-BA. \tag{2}$$

Yukawa has shown that the "propagation" equations of such a field can be written in the case of a scalar meson field, for instance, in the form

$$[[U, p^{\mu}], p_{\mu}]-x^2U=0. \tag{3}$$

The generality of a non-local field being much greater than that of an ordinary field, this equation alone leaves too much arbitrariness, and some supplementary equations are needed. Born's principle of reciprocity can be used here as a guide and leads to the two equations

$$(Y)\begin{cases}\xi(U)\equiv[[U, x^{\mu}], x_{\mu}]+\lambda^2U=0,\\ \eta(U)\equiv[[U, x^{\mu}], p_{\mu}]\equiv[[U, p_{\mu}], x^{\mu}]=0,\end{cases} \tag{4}$$

in which λ is a new fundamental constant having the dimension of a length. Following Yukawa, the effect of equations (4) can be made clear by using the representation of the x's and p's in which the x's are diagonal. The field U is then described by its matrix elements $(x'|U|x'')$, and, as is readily seen, the equations (4) involve that these matrix elements vanish unless the quantities $r^{\mu}=x'^{\mu}-x''^{\mu}$ are of the order of magnitude of λ. On the other hand, equation (3) becomes in this representation the ordinary wave equation of a scalar meson field with the variables $X^{\mu}=\frac{1}{2}(x'^{\mu}+x''^{\mu})$. This shows that the X-space is the ordinary space-time continuum, whereas the r-space is a new space describing the "internal structure" of the elementary particles. Equations (4) which define the internal structure of the particles correspond to typically new features of the theory. Equation (3) defines the motion of the particles and is similar to the usual propagation equation. It then seems natural to assume that in a more general theory, in which the interactions are taken into account, the propagation equation (3) should be deduced from a variation principle as in conventional field theory, whereas the equations (Y) generalizing the equations (4) should be obtained by some new principle, such as the reciprocity principle, or, more simply, by invariance considerations. An attempt has been made$^{(2)}$ to develop along these lines a theory of non-local fields including interactions, and we should like in this note to sketch briefly its main features.

II. Variation principle and conservations.

We suppose that the equations (Y) are known. They must have the property of requiring that the matrix elements $(x'|U^{(i)}|x'')$ vanish unless the r^{μ}'s are of the same order of magnitude as λ. In this way, the field becomes, in the limit $\lambda \to 0$, an ordinary local field, which, in the present notations, reads $u^{(i)}(X)\,\delta(r)$, where $\delta(r)$ is the fourdimensional Dirac function. Let also be given a Lagrange function $\mathfrak{L}$, which is a polynomial in the field variables and the operators p_{μ}, x^{μ}, and which, in the case of a local field, reduces to $L(X)\,\delta(r)$, where $L(X)$ is the ordinary Lagrange function. In conventional field theory, the action integral is defined by $I_{\Omega}=\int_{\Omega}dX\,L(X)$, where Ω is any part of space-time. Clearly, if in the limit $\lambda \to 0$ the theory is to become identical with the usual one, the action integral must be

$$I_{\Omega}=\int_{\Omega}dX\int dr\,\mathfrak{L}(X,r), \tag{5}$$

where $\mathfrak{L}(X,r)$ is the matrix element $(x'|\mathfrak{L}|x'')$ expressed in terms of the variables X^{μ} and r^{μ}. The integral I_{Ω} is real if $\mathfrak{L}$ is Hermitian. It is convenient to introduce the operator J_{Ω} defined by $(x'|J_{\Omega}|x'')=1$ if the point X is inside Ω, and $(x'|J_{\Omega}|x'')=0$ if X is outside Ω. We can then write I_{Ω} as a trace

$$I_{\Omega}=Tr\,J_{\Omega}\mathfrak{L}=Tr\,\mathfrak{L}\,J_{\Omega}=\int dx'\,(x'|J_{\Omega}\,\mathfrak{L}|x'). \tag{6}$$

Considering an arbitrary variation $\delta U^{(i)}$ of the field, the corresponding variation of $\mathfrak{L}$ can be written, for instance in the case in which $\mathfrak{L}$ is a polynomial in $U^{(i)}$ and $[U^{(i)},p_{\mu}]$.

$$\delta\mathfrak{L}=\sum_{m,i,\mu}\mathfrak{L}^{m}_{(i)1}\,\delta U^{(i)}\,\mathfrak{L}^{m}_{(i)2}+\mathfrak{L}^{\mu,m}_{(i)1}\,[\delta U^{(i)},p_{\mu}]\,\mathfrak{L}^{\mu,m}_{(i)2}, \tag{7}$$

in which the $\mathfrak{L}$'s are functions of the field variables generalizing the partial derivatives of the Lagrange function. The variation of the action integral can be written in the form

$$\delta I_\Omega = Tr \sum_i \mathfrak{P}^{(i)} \, \delta U^{(i)} + \int_\Sigma d\sigma_\mu \int d\tau \, \mathfrak{M}^\mu (\delta U^{(i)}), \tag{8}$$

where the second term is a surface integral extended over the boundary Σ of the domain Ω. In equation (8) $\mathfrak{P}_{(i)}$ and $\mathfrak{M}^\mu$ are given by

$$\mathfrak{P}_{(i)} = \sum_{m,\mu} \mathfrak{L}^m_{(i)2} J_\Omega \mathfrak{L}^m_{(i)1} - [\mathfrak{L}^{\mu,m}_{(i)2}, p_\mu] J_\Omega \mathfrak{L}^{\mu,m}_{(i)1} - \mathfrak{L}^{\mu,m}_{(i)2} J_\Omega [\mathfrak{L}^{\mu,m}_{(i)1}, p_\mu], \tag{9}$$

$$\mathfrak{M}^\mu (\delta U^{(i)}) = \sum_{i,m} \mathfrak{L}^{\mu,m}_{(i)1} \, \delta U^{(i)} \, \mathfrak{L}^{\mu,m}_{(i)2}. \tag{10}$$

If the variation $\delta U^{(i)}$ is such that $(x' | \delta U^{(i)} | x'')$ vanishes if x' and x'' are within a neighbourhood of Σ of the order of magnitude of λ, the surface integral in (8) vanishes. Thus, the field equations obtained from the principle that the action integral should be stationary are $\mathfrak{P}_{(i)} = 0$. In order to get field equations consistent with the equations (Y), it may be necessary to restrict the variations $\delta U^{(i)}$ to those allowed by the equations (Y). Lagrangian multipliers $V_{(i)}$ and $W_{(i)}$ have then to be introduced, and the field equations become

$$\mathfrak{P}_{(i)} + \xi(V_{(i)}) + \eta(W_{(i)}) = 0. \tag{11}$$

Conservation equations are easily deduced from equation (8). If the variation $\delta U^{(i)}$ is that due to an infinitesimal change of the frame of reference given by $\bar{x}^\mu = x^\mu - \delta x^\mu$, the corresponding variation δI_Ω can be directly seen to be $\delta I_\Omega = \int_\Sigma d\sigma_\mu \int d\tau \, \delta X^\mu \, \mathfrak{L}$. On the other hand, if the field equations (11) are taken into account, the variation δI_Ω is given by the second term in (8), only. Hence, we get the general conservation equation

$$\int_\Sigma d\sigma_\mu \int d\tau \{ \mathfrak{M}^\mu (\delta U^{(i)}) - \delta X^\mu \, \mathfrak{L} \} = 0. \tag{12}$$

Application of (12) to infinitesimal translations, Lorentz transformations, and gauge transformations, leads to conservation equations corresponding to energy-momentum, angular momentum and electric current-charge, respectively. Note that these conservation equations are obtained in the form of surface integrals extended to Σ, which vanish as a consequence of the field equations.

A remarkable feature of the preceding theory should now be emphasized. It is seen from (9) that the field equations (11) depend explicitly on the volume Ω occurring in the action integral. However, it can be shown that this dependence disappears in the case of free fields. It follows that, in the more general case in which interactions are taken into account, the dependence on Ω will be negligible if Ω is so large that the interactions are negligible on the boundary Σ of Ω, and there will be a limit if Ω increases indefinitely in all directions. This property is sufficient to make the theory consistent. In the conservation equations resulting from (12), the surface Σ cannot be taken arbitrarily; it is then not possible to go over from the integral form of the conservation equations to the differential form of equations expressing that certain tensors have vanishing divergence. From these considerations it follows that in non-local field theory a collision cannot, as in conventional field theory, be considered as a succession of independent infinitesimal processes. A collision process has to be regarded as a whole, and conservation laws hold only between the initial and the final states. Consequently, there is no hope of putting the theory into Hamiltonian form, and of finding an equation corresponding to Schrödinger's; it will be necessary to use a formalism similar to that of the S-matrix theory.

III. Free fields.

Although it is rather trivial, it is important to study in detail the case of free fields, in view of the fact that all treatments of field theory known as yet use perturbation methods, starting from free particles as unperturbed states. The simplest field is the scalar meson field for which the equations (Y) have already been given in (4). As the Lagrange function it is natural to take

$$\mathfrak{L}=[U^*,p^\mu][U,p_\mu]+\varkappa^2U^*U. \tag{13}$$

The variation principle gives the field equation

$$\{[[U,p_\mu],p^\mu]-\varkappa^2U\}J_\Omega=\xi(V)+\eta(W), \tag{14}$$

and the conjugate of the latter. The methods discussed in the preceding paragraph also lead to definitions of energy-momentum, angular momentum and electric current-charge. It can be shown that the general solution of equations (4) and (14) is given in the case in which Ω is the whole space-time by the expansion

$$U=\int dk\,dl\,u(k,l)\,e^{ikx}\,e^{ilp}, \tag{15}$$

(we write kx for the scalar product $k_\mu x^\mu$ etc...), where

$$u(k,l)=u_0(k)\delta(k^2+\varkappa^2)\delta(kl)\delta(l^2-\lambda^2)+u_1(k,l)\delta(kl)\delta(l^2-\lambda^2), \tag{16}$$

where $u_1(k,l)$ satisfies the condition

$$\int dl\,u_1(k,l)\delta(kl)\delta(l^2-\lambda^2)=0. \tag{17}$$

It is readily seen that the part $u_1(k,l)$ of the field does not give any contribution to energy-momentum and electric charge. This suggests that there is still an equation (Y) missing, and that we should add

$$u_1(k,l)=0, \tag{18}$$

an equation which is clearly Lorentz invariant and compatible with the other equations. When this supplementary equation is taken into account, equation (14) becomes equivalent to equation (3) originally given by Yukawa. Further more, the number of degrees of freedom of the field becomes equal to that of the corresponding usual local field, and the whole theory is equivalent to the conventional field theory in the large, i.e. when averages are taken over regions of dimensions much larger than λ. This seems very satisfactory in view of the fact that conventional field theory provides a correct description of free particles, difficulties occurring only when the interactions are taken into account.

A non-local theory of the electron can be built along the same lines, starting from the Lagrange function

$$\left.\begin{aligned} 2\mathfrak{L} &= \frac{1}{2}\{\Psi^{+}\gamma^{\mu}[\Psi, p_{\mu}] - [\Psi^{+}, p_{\mu}]\gamma^{\mu}\Psi\} + \varkappa\Psi^{+}\Psi \\ &+ \frac{1}{2}\{\Psi'^{+}\gamma^{\mu}[\Psi', p_{\mu}] - [\Psi'^{+}, p_{\mu}]\gamma^{\mu}\Psi'\} + \varkappa\Psi'^{+}\Psi', \end{aligned}\right\} \tag{19}$$

in which the adjoint field function Ψ^{+} and the charge conjugate functions Ψ', Ψ'^{+} are related to the non-local field function Ψ in the usual manner. Some arbitrariness arises in the choice of the equations (Y). Yukawa linearizes them according to the principle of reciprocity. This procedure has the disadvantage of introducing a new degree of freedom which seems difficult to interpret. In the present theory the symmetry postulated by the principle of reciprocity is not observed anyway, since the equations (Y) and the propagation equations are treated on a different footing. There is then very little foundation for linearizing the equations (Y), and it seems reasonable to keep them unchanged since this gives a theory of free electrons which after introduction of a supplementary equation similar to (18), is equivalent in the large to the conventional theory.

The second quantization of the fields considered above can be performed without difficulty by defining suitable bracket relations for the field variables.(2)

IV. Electrons interacting with the electromagnetic field.

Although the detailed treatment of this problem has not yet been completed, we should like to suggest here briefly a method which can be used.

The first question which arises is whether the electromagnetic field should also be treated as a non-local field. Here we note that one of the main ideas in non-local field theory is that if λ tends to zero, the field must have an ordinary local field as limit. This requirement cannot be satisfied in the case of a field corresponding to particles with vanishing rest mass. Consider, for instance, a plane wave in X-space; it reads $e^{ikX}f(r)$. By invariance considerations it is seen that $f(r)$ has to be of the form $f(r) = a\delta(r) + g(r^2, kr)$. Now, if we exclude the trivial case in which g vanishes identically as $\lambda \to 0$, the best we can hope is that, in the limit $\lambda \to 0$, the function g vanishes unless $r^2 = 0$, $kr = 0$. But in the case of a vanishing rest-mass, $k^2 = 0$, and we see that g will be different from zero for any r parallel to k. Hence, we must have $g \equiv 0$, and the field reduces to a local field.

The electromagnetic field will be described by its four-vector potential $A_\mu(X)$. The way it has to be coupled with the non-local electron field is entirely defined by the requirement of gauge invariance. A gauge transformation of the potential reads

$$A_\mu(X) \to A_\mu(X) + \frac{1}{\epsilon} \frac{\partial \alpha(X)}{\partial X^\mu}. \tag{20}$$

For the electron field functions the transformation is

$$\left.\begin{aligned} &\Psi \to e^{i\alpha}\,\Psi, \ \ \Psi^+ \to \Psi^+\, e^{-i\alpha}, \\ &\Psi' \to e^{-i\alpha}\,\Psi', \ \ \Psi'^+ \to \Psi'^+\, e^{i\alpha}. \end{aligned}\right\} \tag{21}$$

Note that in order to obtain the exponential factors at the right place in the transformation formulas of the charge conjugate field functions these must be defined by means of the transposed field functions: $\Psi' = C\tilde{\Psi}^+$, etc......, where $(x'|\tilde{\Psi}^+|x'') = (x''|\Psi^+|x')$. It can now easily be seen that a gauge invariant Lagrange function is obtained from (19) by the following substitutions

$$\left.\begin{aligned} &[\Psi, p_\mu] \to [\Psi, p_\mu] - i\epsilon A_\mu \Psi, \\ &[\Psi^+, p_\mu] \to [\Psi^+, p_\mu] + i\epsilon \Psi^+ A_\mu, \end{aligned}\right\} \tag{22}$$

and for the charge conjugate functions, similar formulas with ϵ replaced by $-\epsilon$. The same substitutions, of course, have to be performed in the equations (Y), and this gives a new kind of interaction which has no equivalent in conventional field theory.

From the Lagrange function, the field equations are obtained by a straightforward procedure. The problem is then to solve and to quantize these equations. In recent years, several nearly equivalent treatments of quantum electrodynamics have been formulated. The first covariant theory is due to Tomonaga(3), and independently to Schwinger(4). However, as the Schrödinger equation plays an important part in this formalism, it does not seem easy to use a similar treatment in non-local field theory. In Feynman's formulation(5), on the other hand, the starting point is essentially an expression of the solutions of the field equations in terms of initial values given on a certain space-like surface. In the present theory, the non-local character of the field makes it difficult to formulate the way in which initial values on a given surface should be specified. Very recently, an alternative formulation of quantum electrodynamics has been given by Yang and Feldman(6), and by Källen(7). It has the advantage of giving the S-matrix very directly. Thus, it seems to be the most suitable method for an extension to non-local fields.

The equations (Y) and the equations obtained from the variation principle are of the form

$$\left.\begin{aligned} \{\gamma^\mu [\Psi, p_\mu] + \varkappa\Psi\} J_\Omega &= i\epsilon\gamma^\mu A_\mu \Psi J_\Omega, \\ \Box A_\mu &= -J_\mu, \\ \xi(\Psi) &= 0, \\ \eta(\Psi) &= i\epsilon A_\mu [\Psi, x^\mu], \text{ etc...,} \end{aligned}\right\} \tag{23}$$

where J_μ is the four-vector current of the electrons. For the sake of simplicity, we have not written the terms containing the Lagrangian multipliers. Consider first the second of the equations (23). If the right-hand side J_μ is supposed to be a given function in space-time, the general solution of the equation is obtained by adding the general solution of the corresponding homogeneous equation to a special solution of the complete equation. This special solution can be expressed, as is well-known, by means of a Green function $D(X)$ in the form of an integral $\int D(X-X') J_\mu(X') dX'$. There is some arbitrariness in the choice of the Green function, which is removed if one defines boundary values for instance, for the time X^4 going to $-\infty$, or to $+\infty$. Let us call $D_{\text{ret}}(X)$ and $D_{\text{adv}}(X)$ the Green functions which tend to zero if $X^4 \to -\infty$ and $+\infty$, respectively. These functions are the usual retarded and advanced potentials. Then we have

$$A_\mu(X_{(1)}) = A_\mu^{\text{in}}(X_{(1)}) - \int D_{\text{ret}}(X_{(1)} - X_{(2)}) J_\mu(X_{(2)})\, dX_{(2)}, \tag{24}$$

$$A_\mu(X_{(1)}) = A_\mu^{\text{out}}(X_{(1)}) - \int D_{\text{adv}}(X_{(1)} - X_{(2)}) J_\mu(X_{(2)})\, dX_{(2)}, \tag{24'}$$

where the fields $A_\mu^{\text{in}}(X)$ and $A_\mu^{\text{out}}(X)$ are solutions of the free field equations. They represent in fact the incoming and the outgoing fields, respectively, since the integrals in (24) and (24′) vanish if $X^4 \to -\infty$ or $+\infty$, respectively. A very similar treatment can be applied to the other equations although they are non-local. It leads to

$$\Psi(1) = \Psi^{\text{in}}(1) + \epsilon\, Tr_{(2)}\, S_{\text{ret}}^\mu(1,2)\, A_\mu(2)\, \Psi(2), \tag{25}$$

$$\Psi(1) = \Psi^{\text{out}}(1) + \epsilon\, Tr_{(2)}\, S_{\text{adv}}^\mu(1,2)\, A_\mu(2)\, \Psi(2), \tag{25'}$$

where 1 and 2 stand for two different sets of variables x^μ, p_μ commuting with each other; Tr refers to the trace operation defined in equation (6). The Green functions S^μ depend on these two sets, which means that their matrix elements are functions of four points in space-time. Now, if we give to the right-hand sides of the equations (24) and (25) their actual values in terms of the field functions, we get a system of integral equations equivalent to the equations (23), which furthermore includes the boundary conditions for $X^4 \to -\infty$. The same holds for (24′) and (25′) with boundary conditions for $X^4 \to +\infty$.

With this new form of the equations the second quantization can easily be performed. In conventional local field theory, it can be shown(6) that, as a consequence of the usual bracket relations of the field quantities, the incoming and the outgoing fields satisfy free field bracket relations. In conventional field theory, the starting point is the bracket relations of two field quantities, which are simple only if the two quantities are taken at points lying outside each other's light cones. The corresponding notions cannot easily be introduced in non-local field theory, since a field quantity depends on two points in space-time. It then seems natural to define the quantization by postulating free field bracket relations for the incoming fields, for instance. The bracket relations of the field quantities themselves and of the outgoing field functions are consequences of the latter. When it has been shown that the outgoing field satisfies the same free field bracket relations as the incoming field, it follows from a well-known theorem that there exists a matrix S such that

$$\Psi^{\text{out}}=S^{-1}\,\Psi^{\text{in}}\,S, \qquad A_\mu^{\text{out}}=S^{-1}\,A_\mu^{\text{in}}\,S, \tag{26}$$

it is clearly the collision matrix. The equations (24), (24′), (25), (25′) and (26) leading to the S-matrix can be solved by means of series expansions in powers of the coupling constant.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Force Potentials in Quantum Field Theory*

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(Received July 31, 1950)

1. Introduction and Summary

As early as 1935 Yukawa conjectured that the nuclear force, which ties together the component nucleons into a solid nucleus, could be attributed to an intermediate field with an intrinsic mass corresponding to the range of the nuclear force. The general success of the meson theory that followed the discovery of such particles in cosmic rays, has been so great that one cannot now discuss the nuclear and cosmic phenomena without the help of Yukawa's idea on the nature of mesons. One must admit, however, that the meson theory in the present stage is far from satisfactory almost in every detail regarding its quantitative predictions on various phenomena. This may be due to our insufficient knowledge about the nature of the real mesons as well as to a more profound crisis of the present quantum field theory which precludes us from drawing reasonable conclusions out of physical assumptions. Both difficulties are intrinsically connected with each other and their complete solution does not seem to be an easy one.

Here we will pick up one characteristic feature of the Yukawa theory, i.e. the theory of nuclear forces. Up to now, however, there is a wide gap between the nuclear potentials predicted on the basis of various meson theories and the one that is obtained empirically. No simple assumptions have ever succeeded in explaining the quantitative behavior of the internucleonic interactions. The theoretical side of the difficulty seems to consist of two facts. One is the high singularity of the nuclear potential which is more or less of a common origin in the present quantum theory, while the other is that we have no established formalism of the relativistic many-body problem, a peculiar situation encountered in the theory of the deuteron. At least formally, the *S*-matrix theory of Heisenberg affords a means to deal equally with scattering processes and bound systems. But the latter case is more helpless since the higher order terms in the coupling constant play an essential role in the determination of energy levels. In the non-relativistic limit, such higher order effects were properly represented in the interaction potential appearing in the Schrödinger eigenvalue problem, a typical case being that of the hydrogen atom. At the genesis of the meson theory, Yukawa borrowed this successful idea from the electromagnetic theory. There are, however, some remarkable distinctions between the electromagnetic and meson field. The latter has mass and charge, together with strong coupling constants. The Coulomb potential, being not quantized, was an exact theoretical consequence, while there is no such classical potential in the mesonic case. These situations exhibit themselves in the relativistic effect, or the retardation on one hand, and in the quantum effect, or the recoil and higher order forces on the other. Before deciding between the existing " korrespondenzmässigen " theories and any more revolutionary ones, we probably have to re-examine the old concept of potential and improve it so as to conform more closely to the rigorous field theoretical view.

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* Preliminary report: Prog. Theor. Phys. 5 (1950), 321. Some errors are corrected in the text.

In the following sections we apply the recent method of quantum electrodynamics to the analysis of these problems. At the present stage, however, we have to abandon the complete relativistic covariance, which is one of the most beautiful achievements of the new theory, but rather confine ourselves to more crude and approximate considerations. In Section 2 we begin with the derivation of the second order potential with the aid of the covariant field theory, and point out the ambiguities that naturally arise. The latter leads to the consideration of the fourth order potential in the next section. An extension of these methods to still higher order terms seems almost impracticable, but very simplified arguments are presented in the last section. The examination, however, of the existing individual meson models in the light of the obtained results will be carried out only on a later occasion.

2. The Second Order Potential

The derivation of the second order interaction potential between two Fermi particles has been carried out in various ways. Originally the Coulomb potential for charged particles was directly imported from the classical theory. Its relativistic correction was first derived by Breit[1] also on the basis of the correspondence principle. He afterwards developed his principle of the approximate relativistic covariance[2] to obtain some general expressions for electromagnetic as well as nuclear potentials with relativistic corrections. On the other hand, perturbation calculations based on the quantized field theory were carried out by many authors[3], with the results that were essentially the same as that expected from the correspondence arguments. In the meson theory an attempt was also made[4] to use canonical transformations instead of the ordinary perturbation method. Very recently this was replaced by the more elegant and perfect theory of Tomonaga and Schwinger[5], from which standpoint some papers have already been published on the meson potentials[6]. All these results seem to be in general, but not complete, agreement. The correct interpretation of them will only be given by a careful inspection and reflection on the meaning of the potential.

Here we start from the Tomonaga-Schwinger equation (in natural units):

$$i\frac{\delta}{\delta\sigma}\Psi[\sigma]=H(x)\Psi[\sigma], \tag{1}$$

$$H(x)=-\varepsilon j_\lambda \varphi_\lambda \tag{2}$$

for the interaction of a fermion field ψ with a boson field φ. ε is the coupling constant, and j_λ a quantity bilinear in $\bar\psi$ and ψ. λ represents the concurrent tensor and isotopic spin indices. By the routine transformation[1]

$$\Psi(\sigma)=\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty}H(x')\,\varepsilon(\sigma\sigma')\,(dx')\right]\Psi_1[\sigma], \tag{3}$$

we obtain

$$i\frac{\delta}{\delta\sigma}\Psi_1[\sigma]=-\frac{1}{4}\,i\left[H(x),\int_{-\infty}^{\infty}H(x')\,\varepsilon(\sigma\sigma')\,(dx')\right]\Psi_1[\sigma] \tag{4}$$

to the second order in ε. For a system in which real bosons are absent, this gives

$$i\frac{\delta}{\delta\sigma}\Psi_1[\sigma]=\left[-\frac{1}{2}\varepsilon^2 j_\lambda(x)\int_{-\infty}^{\infty}j_\mu(x')\bar{\Delta}_{\lambda\mu}(x-x')\,(dx')\right]\Psi_1[\sigma]=H^{(2)}(x)\Psi_1[\sigma]$$

using the commutation relation

$$\begin{gathered}[\varphi_\lambda(x),\varphi_\mu(x')]=i\Delta_{\lambda\mu}(x-x')=iO_{\lambda\mu}\Delta(x-x'),\\ \bar{\Delta}_{\lambda\mu}(x)=-\frac{1}{2}\varepsilon(x)\Delta_{\lambda\mu}(x)\ \text{or}\ -\frac{1}{2}O_{\lambda\mu}(\varepsilon(x)\Delta(x)),\end{gathered} \tag{6}$$

where $\Delta_{\lambda\mu}(x)$ contains the essential factor $\Delta(x)$ as defined by Schwinger. In order to get the usual potential form we make use of the Fourier decomposition of $\bar{\Delta}$, or more conveniently the operational relation

$$\begin{gathered}\int_{-\infty}^{\infty}\bar{\Delta}(x)f(x)\,dx_0=\frac{1}{4\pi}\frac{1}{r}\exp\left(-r\sqrt{\mu^2+(d/dx_0)^2}\right)f(x)\Big|_{x_0=0},\\ r^2=\sum_{i=1}^{3}x_i^2,\qquad x_0=t,\end{gathered} \tag{7}$$

where μ characterizes the mass of the field. In case $\mu=0$,

$$\begin{aligned}\int\bar{D}(x)f(x)\,dx_0&=\frac{1}{4\pi}\frac{1}{2r}\left(\exp\left(-r\frac{d}{dx_0}\right)+\exp\left(r\frac{d}{dx_0}\right)\right)f(x)\Big|_{x_0=0}\\ &=\frac{1}{4\pi}\frac{1}{2r}\left(f(\boldsymbol{r},t-r)+f(\boldsymbol{r},t+r)\right)\\ &=\frac{1}{4\pi}\frac{1}{2r}\left(f(\boldsymbol{r})_{\text{ret}}+f(\boldsymbol{r})_{\text{adv}}\right).\end{aligned} \tag{8}$$

We should apply these formulas after making the space-like surface σ flat. Thus in Eq. (5) the interaction becomes

$$H^{(2)}(x)=-\frac{1}{2}\frac{\varepsilon^2}{4\pi}j_\lambda(\boldsymbol{r})\int O_{\lambda\mu}\,V(\boldsymbol{r}-\boldsymbol{r}')j_\mu(\boldsymbol{r}')d\boldsymbol{r}', \tag{9}$$

$$V(\boldsymbol{r}-\boldsymbol{r}')=\begin{cases}\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\exp\left(-|\boldsymbol{r}-\boldsymbol{r}'|\sqrt{\mu^2+(d/dt)^2}\right), \quad \text{or} \\ \dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\dfrac{1}{2}\left\{\exp\left(-|\boldsymbol{r}-\boldsymbol{r}'|\dfrac{d}{dt}\right)+\exp\left(|\boldsymbol{r}-\boldsymbol{r}'|\dfrac{d}{dt}\right)\right\}.\end{cases} \tag{10}$$

$H^{(2)}(x)$ means the potential energy density for the particle at x due to other particles. In the two-particle system in which we are interested, the total potential energy will be written down in configuration space as

$$H^{(2)}(12)=-\frac{1}{2}\frac{\varepsilon^2}{4\pi}\{j_\lambda(1)\,O_{\lambda\mu}V(12)j_\mu(2)+j_\mu(2)\,O_{\mu\lambda}V(21)j_\lambda(1)\}, \tag{11}$$

where 1, 2 refer to the numbering of the particles, and j_λ means the quantity introduced before, but represented in the configuration space. V is an operator operating on the quantity that comes after it. We may regard (11) as the potential energy to be inserted in the ordinary Schrödinger equation, in which j_λ is a constant operator and d/dt is replaced by

$$\frac{d}{dt_i}=i\,[H_i^{(0)},\ \ ], \qquad i=1,2, \tag{12}$$

$H^{(0)}$ being the free state Hamiltonian for each particle. We note here that the r-dependent potential itself should always be thrown under the operation (12). Eq. (10) tells us that V corresponds to the attached field introduced by Dirac and others[8]. Its appearance is a consequence of the Schwinger transformation (3). When the real boson field is absent (no incoming and outgoing waves), the attached field is equivalent to the retarded field which follows from the transformation

$$\Psi[\sigma]=\exp\left[-i\int_{-\infty}^{\sigma}H(x')\,(dx')\right]\Psi_1[\sigma]. \tag{13}$$

Though they should be equivalent under the above condition, the form of the potential derived from these transformations is apparently not equivalent. Since such a condition is eliminated in the Schrödinger equation, we here meet with an ambiguity. On physical grounds, however, the symmetrical form may be preferred to the retarded form which is not invariant under the interchange of past and future. It is to be noted that the ambiguity does not arise in the non-relativistic limit for which the familiar potential $1/r$ or $\exp(-\mu r)/r$ results.

Instead of using the rather unesthetical one-time theory to derive the potential, we may also invoke the formula[11]

$$\psi(x)=\int_\sigma S(x-x')\,\gamma_\mu\,\psi(x')\,d\sigma'_\mu,$$
$$\bar\psi(x)=\int_\sigma d\sigma_\mu'\,\bar\psi(x')\,\gamma_\mu\,S(x'-x), \tag{14}$$

with
$$\frac{1}{i}\,S(x-x')=\{\psi(x),\ \bar\psi(x')\},$$

and express the retarded and advanced quantities $j_\lambda(x)$ in terms of ψ and $\bar\psi$ on the surface σ. The resulting interaction is non-local, or in other words, velocity dependent. The Schrödinger equation with such an interaction is then an integro-differential equation, and its solution will never be an easy task. The non-local effect, however, will set in only at very close distances for which large momentum transfers between the particles become important. Thus if we will take account of the retardation in an approximate manner, we may usually be allowed to expand (10) in powers of d/dt and retain only the first few terms. But if j_λ contains a Zitterbewegung (or non-static) part, we cannot do this, for it means the transition from positive to negative state or vice versa, and $d/dt\sim\pm 2mi$, where m is the mass of the particle. If $2m$ is much larger than μ, (10) gives a rapidly oscillating function of the form $\cos(2mr)/r$ which had rather better be substituted by a delta function:

$$\frac{1}{4\pi}\,\frac{1}{r}\cos 2mr\sim-\frac{1}{4m^2}\Big(1-\frac{1}{4m^2}\,\Delta+\cdots\Big)\,\delta(\mathbf{r}). \tag{15}$$

The problem of ambiguity mentioned above, however, is not yet settled at all. We show this with the example of the Breit interaction. In this case,

$$H(x)=-\varepsilon j_\lambda A_\lambda,\qquad j_\lambda=i\bar\psi\gamma_\lambda\psi, \tag{16}$$

and

$$H^{(2)}(x)=-\frac{1}{2}\,\frac{\varepsilon^2}{4\pi}\int j_\lambda(x)\,\frac{\cos\left(\frac{d}{dt'}\right)}{r}\,j_\lambda(x')\,d\mathbf{r}'. \tag{17}$$

This is the Møller formula.[3] The fourth component of the product $j_\lambda j_\lambda'$ corresponds to the Coulomb force with small retardation effect. An expansion in d/dt, which is allowed here, yields the potential in configuration space

$$H^{(2)}(12)=\frac{\varepsilon^2}{4\pi}\left[(\gamma_\lambda)_1\,\frac{1}{r}\,(\gamma_\lambda)_2+\frac{1}{4}\left(\frac{d^2}{dt_1^2}+\frac{d^2}{dt_2^2}\right)(\gamma_4)_1\,r\,(\gamma_4)_2\right]+\cdots \tag{18}$$

When this is evaluated, we do not obtain the Breit formula, contrary to our expectation! However, (18) may also be written as

$$H^{(2)}(12)=\frac{\varepsilon^2}{4\pi}\left[(\gamma_\lambda)_1\,\frac{1}{r}\,(\gamma_\lambda)_2-\frac{1}{2}\,\frac{d}{dt_1}\,\frac{d}{dt_2}\,(\gamma_4)_1\,r\,(\gamma_4)_2\right], \tag{19}$$

for the energy conservation means

$$\frac{d}{dt_1}=-\frac{d}{dt_2} \tag{20}$$

neglecting the interaction term which amounts to a higher order correction. From (19) now follows the desired formula :*

$$H^{(2)}(12)=\left[(\gamma_4)_1(\gamma_4)_2-\frac{1}{2}\left\{(\gamma_i)_1(\gamma_i)_2+\frac{((\gamma_i)_1,\boldsymbol{r})((\gamma_i)_2,\boldsymbol{r})}{r^2}\right\}\right]\frac{1}{r}. \tag{21}$$

The assumption (20) was also made by Bethe and Fermi[3], and more explicitly by Toyoda[9] in meson theory. It is true that an ambiguity does not influence the second order S-matrix corresponding to the elastic scattering of two particles, since in this case (20) holds exactly. But a potential in its proper sense should contain those matrix elements in which energy is not conserved and which brings the system into a virtual state. From the above result we see that the potential is indeterminate to within an arbitrary matrix whose diagonal elements with respect to energy are zero. Let us try to find out how this problem can be settled.

We make here a requirement that the two-body problem shall be rendered to a relativistic form as far as possible. Thus we assume a pair of the Dirac many-time equations for the wave function $\psi(x_1, x_2)$ describing a two-particle system under action at a distance :

$$\begin{aligned}\left\{i\frac{\partial}{\partial t_1}-H_1-V(12)\right\}\psi=0,\\ \left\{i\frac{\partial}{\partial t_2}-H_2-V(21)\right\}\psi=0.\end{aligned} \tag{22}$$

H_1 and H_2 being the Hamiltonians for free state. In general the potentials $V(12)$ and $V(21)$ may be different. By a standard transformation (22) can be brought to the interaction representation

$$\begin{aligned}\left\{i\frac{\partial}{\partial n_1}-V(12)\right\}\psi=0,\\ \left\{i\frac{\partial}{\partial n_2}-V(21)\right\}\psi=0.\end{aligned} \tag{23}$$

where the time-like parameters n may point to different directions, but for the time being assumed to be parallel. Now an integrability condition should be enounced :

$$i\left(\frac{\partial V(21)}{\partial n_1}-\frac{\partial V(12)}{\partial n_2}\right)=[V(12), V(21)]. \tag{24}$$

---

* An alternative method to derive (21) is to separate the pure Coulomb interaction beforehand and treat only the transversal part in the above way.

Suppose the potentials V to be defined in terms of the variables of the two particles which stand in a definite relation to each other with respect to their space-time position, thus taking account of the retardation effect. Then, since the relative position of the particles should remain invariant in V and n_1 is parallel to n_2, we have

$$\frac{\partial V}{\partial n_1}=\frac{\partial V}{\partial n_2}, \tag{25}$$

so that

$$i\frac{\partial}{\partial n}(V(21)-V(12))=[V(12),\, V(21)], \tag{26}$$

which in turn requires that

$$V(12)=V(12), \tag{27}$$

as will be seen by developing V in a power series in the coupling constant. (27) may be regarded as expressing Newton's third law of motion. Let us test this condition for the above mentioned example. The Møller interaction gives, in view of (17) and (18),

$$\begin{aligned} V(12)\,|_{t_1=t_2}&=\frac{\varepsilon^2}{4\pi}\cdot\frac{1}{2}\Big\{(\gamma_\lambda)_1\frac{1}{r}(\gamma_\lambda)_2+\frac{1}{2}(\gamma_\lambda)_1\frac{d^2}{dt_2^2}r(\gamma_4)_2\Big\},\\ V(21)\,|_{t_1=t_2}&=\frac{\varepsilon^2}{4\pi}\cdot\frac{1}{2}\Big\{(\gamma_\lambda)_1\frac{1}{r}(\gamma_\lambda)_2+\frac{1}{2}(\gamma_4)_2\frac{d^2}{dt_1^2}r(\gamma_4)_1\Big\}, \end{aligned} \tag{28}$$

hence clearly $V(12)\neq V(21)$! To make the equation of motion (23) integrable, we must therefore supplement the potentials by some additional terms. This can be performed by a transformation independent of past history:

$$\psi=\exp\Big[\frac{i}{4}\,\frac{\varepsilon^2}{4\pi}\Big(\frac{d}{dt_1}+\frac{d}{dt_2}\Big)(\gamma_4)_1\,r\,(\gamma_4)_2\Big]\psi', \tag{29}$$

with the result

$$V''(12)=V''(21)=\frac{\varepsilon^2}{4\pi}\,\frac{1}{2}\Big[(\gamma_\lambda)_1\frac{1}{r}(\gamma_\lambda)_2-\frac{1}{2}\,\frac{d}{dt_1}\,\frac{d}{dt_2}(\gamma_4)_1\,r\,(\gamma_4)_2\Big]+O(\varepsilon^4), \tag{30}$$

which just leads to (21).

The meaning of the above condition may be illustrated as follows. Let us describe the retarded interactions V by the Feynman diagram. So long as these interactions occur successively with sufficiently large intervals of time, there is nothing difficult. But when they come nearer and cross each other, there arises a difference in the order of emission and absorption of quanta represented by the process $V(21)\,V(12)$ or $V(12)\,V(21)$ and the one corresponding to the actual process, which always occurs in the chronological order. Such a situation is characteristic of the relativistic interaction, for the range of the time interval in which the crossing of the lines prevails will tend to zero as $c \to \infty$. In the expression of the S-matrix, however, these discrepancies are properly compensated, giving the correct result in the fourth order term. Thus we may suppose that in amending the second order potential so as to satisfy the condition (27) we are automatically taking account, at least partially, of the higher order interactions. The ambiguities in the second order potential will then be reflected in the higher order ones in such a way that the overall effect turns out identical. In other words, they will correspond simply to our freedom of canonical transformation of the whole representation.

Anticipating the justification of the above considerations in later sections, we shall next derive an alternative form of the potential which seems more convenient for our purpose. From the point of view of the S-matrix, the transformation (3) is a rather clumsy procedure. We had better write down the S-matrix after Dyson[10)] as

$$S=1+\int H(x)\,dx+\frac{1}{2!}\iint P(H(x_1),H(x_2))\,(dx_1)\,(dx_2)+\cdots, \tag{31}$$

and identify its sub-matrix corresponding to the two-particle system with

$$1+\int V(x_1,x_2)\,(dX)+\frac{1}{2!}\iint P(V(x_1,x_2),V(x_1',x_2'))\,(dX)\,(dX)'+\cdots, \tag{32}$$

where X means a set of coordinates describing the motion of the system as a whole, and the symbol P is defined with respect to these coordinates. Indeed, if a relativistic two-body problem could actually be formulated, we could consider a generalization of the center of gravity to which belongs the constants of motion, or the momentum-energy vector. In case the two particles are identical, it is natural to adopt for X

$$X_\mu=\frac{1}{2}\,((x_\mu)_1+(x_\mu)_2), \tag{33}$$

the remaining variables being the relative coordinates

$$x_\mu=(x_\mu)_1-(x_\mu)_2, \tag{34}$$

so that

$$x_1=X+\frac{x}{2}, \qquad x_2=X-\frac{x}{2}. \tag{35}$$

In terms of these variables V becomes, to the second order,

$$H^{(2)}\equiv V(X)=\frac{1}{2}\int P\left(H\left(X+\frac{x}{2}\right),H\left(X-\frac{x}{2}\right)\right)(dx). \tag{36}$$

With the expression (2), it is

$$H^{(2)}=-\frac{i}{4}\varepsilon^2\int j_\lambda\left(X+\frac{x}{2}\right)\Delta_{F\lambda\mu}(x)j_\mu\left(X-\frac{x}{2}\right)(dx),$$
$$\Delta_F(x)=\Delta^{(1)}(x)-2i\bar{\Delta}(x). \tag{37}$$

The appearance of the function $\Delta_F(x)$ as the vacuum expectation value is a common feature of the Feynman-Dyson theory.[5)10)] This makes, at first glance, the potential function non-Hermitic, but actually the term with $\Delta^{(1)}(x)$ does not contribute to the process and may be omitted. In case the retardation expansion is allowed, (37) gives

$$\begin{aligned}H^{(2)}=&-\frac{\varepsilon^2}{4\pi}\Big[j_\lambda(\boldsymbol{r}_1)O_{\lambda\mu}\frac{e^{-\mu r}}{r}j_\mu(\boldsymbol{r}_2)\\&-\frac{1}{2\mu}\left(\frac{d}{dx_0}\right)^2\left\{j_\lambda\left(\boldsymbol{r}_1,X_0+\frac{x_0}{2}\right)O_{\lambda\mu}e^{-\mu r}j_\mu\left(\boldsymbol{r}_2,X_0-\frac{x_0}{2}\right)\right\}_{x_0=0}\Big]\\=&-\frac{\varepsilon^2}{4\pi}\Big[j_\lambda(1)O_{\lambda\mu}\frac{e^{-\mu r}}{r}j_\mu(2)\\&-\frac{1}{2\mu}\left\{\frac{1}{4}\left(\frac{d}{dt_1}\right)^2-\frac{1}{2}\frac{d}{dt_1}\frac{d}{dt_2}+\frac{1}{4}\left(\frac{d}{dt_2}\right)^2\right\}j_\lambda(1)e^{-\mu r}j_\mu(2)O_{\lambda\mu}\Big].\end{aligned} \tag{38}$$

For $\mu=0$, replace $-\exp(-\mu r)/\mu$ by r. This formula applies, being an expansion in $d/dt_1-d/dt_2=i[H_1-H_2,\quad]$, to those transitions in which the two particles possess the same energy sign both in the initial and final state, a wider condition than the previous one. It is to be noted, however, that in this way the separate derivation of $V(12)$ and $V(21)$ in (23) is impossible, and the physical insight into the condition on V is accordingly obscured.

3. The Fourth Order Potential

As was shown in the preceding analysis, we cannot give in general an unambiguous second order interaction unless the fourth order one is specified. They are related to each other in such a way that the second and fourth order (and possibly higher order) terms in the S-matrix come out correctly as the combined effect of these potentials. In other words, the fourth order terms of the S-matrix are the sum of the first order effect of the fourth order potential and the second order effect of the second order potential. In the S-matrix theory a similar circumstance arises when S is expressed as[11)]

$$S=\frac{1-iK}{1+iK}. \tag{39}$$

Expanding S and K:

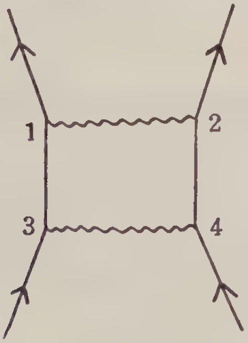
$$S=1+S_2+S_4+\cdots,\qquad K=K_2+K_4+\cdots, \tag{40}$$

we see that

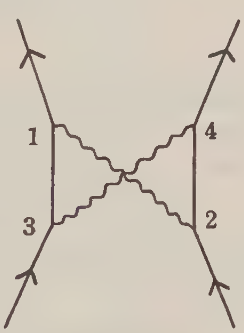
$$S_2^{+}=-S_2,\quad K_2=\frac{i}{2}S_2,\quad K_4=\frac{i}{2}S_4+i(K_2)^2=\frac{i}{2}S_4-i\left(\frac{S_2}{2}\right)^2. \tag{41}$$

Thus $2K_4$ is the difference of S_4 and a two-fold iteration of S_2, and is the real (Hermitic) part of iS_4. In this case, however, the term $(S_2)^2$ corresponds to the process in which the particles are scattered twice but the energy is conserved in the intermediate state (the damping part). In our case, on the other hand, a second order potential is meant to include the reactive effect as well and consequently the fourth order potential is of a nature different from K.

Now from the standpoint of the Feynman theory, the fourth order processes consist of two main graphs (a) and (b). Other graphs contain self-energy effects, and we shall neglect them for the present purpose, for it only gives a small smearing out effect to the source if divergences can be avoided. On iterating the second order graph, we get a graph corresponding to (a) if the two boson lines do not cross, and a graph similar to (b) if they do. But the latter is not exactly (b), for the order of occurrence of the events at 1, 2, 3 and 4 is not chronological, and it does not give the correct matrix element corresponding to (b). Hence the difference between the true fourth order term and the one arizing from the iteration of the second order term can be written as



(a)



(b)

$$-\frac{1}{8}(-i\varepsilon^2)^2\int j_\mu(1)j_\nu(3)[j_\lambda(4),j_\rho(2)]\,\Delta_{F\mu\rho}(12)\,\Delta_{F\nu\lambda}(34)$$
$$\times\frac{1+\varepsilon(13)}{2}\,\frac{1+\varepsilon(42)}{2}(dx_1)(dx_2)(dx_3)(dx_4), \tag{42}$$

where 1, 3 and 4, 2 are understood to refer to different particles. A more symmetrical form is provided by

$$\frac{1}{32}\varepsilon^4\int\Big\{\frac{1+\varepsilon(13)\varepsilon(42)}{2}[j_\mu(1),j_\nu(3)][j_\lambda(4),j_\rho(2)]$$
$$+\frac{\varepsilon(13)+\varepsilon(42)}{4}(\{j_\mu(1),j_\nu(3)\}[j_\lambda(4),j_\rho(2)]+[j_\mu(1),j_\nu(3)]\{j_\lambda(4),j_\rho(2)\})\Big\}$$
$$\times\Delta_{F\mu\rho}(12)\,\Delta_{F\nu\lambda}(34)(dx_1)(dx_2)(dx_3)(dx_4). \tag{43}$$

(41) and (42) contain the sign function $\varepsilon(x)$ which depends on the surface parametrization. In fact, on writing down the matrix element explicitly, the second term of (43) is seen to contain a factor $\varepsilon(x)\, S^{(1)}(x)$. This shows that the potential energy and accompanying eigenvalue problem do not have in general the ordinary covariant character. As it will not be of much importance to obtain the rigorous form of the potential, we shall be content with rough estimations. For this purpose we specify the time axis and proceed with v/c expansions. Then two cases discriminate themselves. The first is the one in which the quantities j are static so that in an expression like $j_\mu(1)\, j_\nu(3)$ we can take account of the difference of t_1 and t_3 by a simple Taylor expansion in t_1-t_3. If, on the other hand, the j are non-static $(\sim v/c)$, then such an expansion fails because of the large Zitterbewegung frequency, and we consequently face the second case. The cross term, in which one j is static and the other non-static, will be considered afterwards.

a) The static case. We use the expression (43), not explicitly writing down the S-functions for the particles, but going with the j_λ themselves. We make beforehand a remark that the first term in the bracket in (43) gives a zeroth order $(\sim(v/c)^0)$ potential while the second is responsible for the retardation correction to it. Thus in the first term we put $j_\lambda(t)\sim 1$ with respect to the time dependence. The Δ-function is, in Fourier representation,

$$\Delta_F(x)=\frac{-2i}{(2\pi)^4}\int e^{i(kx)}\,\frac{(dk)}{k^2+\mu^2}, \tag{44}$$

with the understanding that μ^2 is furnished with a small negative imaginary part. The first term of Eq. (43), which we call (A), becomes then

$$\begin{aligned}&\frac{1}{32}\,\varepsilon^4\,\frac{(-2i)^2}{(2\pi)^8}\int\cdots\int\frac{1+\varepsilon(13)\,\varepsilon(42)}{2}\,[j_\mu(\boldsymbol{r}_1),j_\nu(\boldsymbol{r}_3)]\,[j_\lambda(\boldsymbol{r}_4),j_\rho(\boldsymbol{r}_2)]\,O_{\mu\rho}O_{\lambda\nu}\\ &\times\frac{1}{k^2+\mu^2}\cdot\frac{1}{k'^2+\mu^2}\cdot e^{-ik_0(t_1-t_2)}\,e^{-ik_0'(t_3-t_4)}\,e^{i(\boldsymbol{k},\,\boldsymbol{r}_1-\boldsymbol{r}_2)}\,e^{i(\boldsymbol{k}',\,\boldsymbol{r}_3-\boldsymbol{r}_4)}\\ &\times(dk)\,(dk')\,dt_1\,dt_2\,dt_3\,dt_4\cdot d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_3\,d\boldsymbol{r}_4.\end{aligned} \tag{45}$$

Making use of the fact that $j_\mu(r)$ and $j_\nu(r')$ commute (for equal t) if $\boldsymbol{r}\neq\boldsymbol{r}'$, we can identify r_1 and r_2 with r_3 and r_4 respectively, and put $r_1-r_2=r_3-r_4=\boldsymbol{r}$, thereby reducing the quadruple integral $\int\!\!\int\!\!\int dr$ to a double one. As regards the integration in t, we invoke the centers of time

$$T=\frac{t_1+t_2+t_4+t_3}{4}=\frac{T'+T''}{2},\quad T'=\frac{t_1+t_2}{2},\quad T''=\frac{t_3+t_4}{2},$$

together with

$$t=T'-T'',\quad t'=t_1-t_2,\quad t''=t_3-t_4, \tag{46}$$

and leave out the integration in T to obtain the potential energy. Further putting

$$t'-t''=u, \qquad t=\frac{1}{2}uv, \tag{47}$$

we find

$$\int\cdots\int\frac{1+\varepsilon(13)\varepsilon(42)}{2}e^{-ik_0(t_1-t_2)-ik_0'(t_3-t_4)}dt_1\cdots dt_4$$

$$=\int dT\int_{-\infty}^{\infty}e^{-i(k_0-k_0')u}\frac{1}{2}|u|du\int_{-1}^{1}dv\cdot 2\pi\delta(k_0+k'_0)$$

$$=-\frac{8}{(k_0-k_0')^2}2\pi\delta(k_0+k_0')\int dT. \tag{48}$$

The potential (A) is given by

$$H_A^{(4)}=\left(\frac{\varepsilon^2}{4\pi}\right)^2\frac{1}{2}\int[j_\mu(\boldsymbol{r}_1),j_\nu(\boldsymbol{r}_1)][j_\lambda(\boldsymbol{r}_2),j_\rho(\boldsymbol{r}_2)]\,O_{\mu\rho}\,O_{\lambda\nu}\,V(\boldsymbol{r}_1-\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2,$$

$$V(\boldsymbol{r})=\frac{i}{16\pi^5}\int\frac{1}{k_0^2}\cdot\frac{e^{-i\boldsymbol{kr}}}{k^2+\mu^2}\cdot\frac{e^{i\boldsymbol{k'r}}}{k'^2+\mu^2}\delta(k_0+k'_0)(dk)(dk'). \tag{49}$$

V can be evaluated, with the above mentioned definition at the poles, as

$$V(\boldsymbol{r})=\frac{i}{16\pi^5}\iint\left[-\pi i\left\{\frac{1}{(k^2+\mu^2)^{3/2}}\frac{1}{k'^2-k^2}+\frac{1}{(k'^2+\mu^2)^{3/2}}\frac{1}{k^2-k'^2}\right\}e^{i\boldsymbol{kr}}\,e^{i\boldsymbol{k'r}}\right]dk\,dk'$$

$$=\frac{i}{16\pi^5}\int(-\pi i)\frac{e^{i\boldsymbol{kr}}}{(k^2+\mu^2)^{3/2}}(-2\pi^2)\frac{\cos kr}{r}\times 2\,dk$$

$$=\frac{1}{2\pi r^2}\int\frac{\sin 2kr}{(k^2+\mu^2)^{3/2}}k\,dk=\frac{1}{\pi r}K_0(2\mu r) \tag{50}$$

where K_ν is the modified Bessel function of order ν:

$$K_\nu(x)=\frac{1}{2}\pi i e^{-\frac{1}{2}\pi i\nu}H_{-\nu}^{(1)}(ix). \tag{51}$$

Asymptotically, V behaves as

$$\begin{aligned}V(r)&\sim-(1/\pi r)\ln(\gamma\mu r),\quad \gamma=1.781 \text{ for } 2\mu r\ll 1\\ &\sim(1/2r)(1/\pi\mu r)^{1/2}e^{-2\mu r}\quad \text{for } 2\mu r\gg 1.\end{aligned} \tag{52}$$

The range of the force is thus half of that in the second order potential, corresponding to the fact that two mesons are interchanged between the particles.

The retardation correction has been neglected in the above calculation, but it can be shown that the $(v/c)^1$-effect is zero. In view of the approximate nature of the calcultion, we shall not investigate the $(v/c)^2$ -correction to the fourth order potential which is already small compared to the second order one

On the other hand, the second term of Eq. (43), called (B), gives a $(v/c)^1$-effect. We accordingly put

$$j(1)\,j(3)\sim e^{i\Delta p_0(1)t_1}\cdot e^{i\Delta p_0(3)t_3}. \tag{53}$$

An integral of the form

$$\int\frac{\varepsilon(13)+\varepsilon(42)}{4}\,e^{-i(k_0-k'_0)\frac{t_1-t_3-t_2+t_4}{2}}\,e^{ip(t_1-t_3)}\,e^{iq(t_4-t_2)}\,dt_1\ldots dt_4 \tag{54}$$

is, by a similar procedure, found to be

$$=-\frac{\pi i}{2}\,\frac{1}{p-q}\,\{\delta(k_0-k_0'-2p)-\delta(k_0-k_0'-2q)\}. \tag{55}$$

Using this, the part (B) of the potential can be calculated, giving

$$\begin{aligned}H_B^{(4)}=\frac{i}{16}\left(\frac{\varepsilon^2}{4\pi}\right)^2\int\Big\{\Big[\frac{d}{dt}\,(j_\mu(\boldsymbol{r}_1)\,j_\lambda(\boldsymbol{r}_2)),\,j_\nu(\boldsymbol{r}_1)\,j_\rho(\boldsymbol{r}_2)\Big]-\Big[j_\mu(\boldsymbol{r}_1)j_\lambda(\boldsymbol{r}_2),\\ \frac{d}{dt}\,(j_\nu(\boldsymbol{r}_1)j_\rho(\boldsymbol{r}_2))\Big]\Big\}\,O_{\mu\rho}\,O_{\nu\lambda}\,\frac{e^{-\mu r}}{r}\,\frac{e^{-\mu r}}{2\mu}\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2\\ =\frac{i}{8}\left(\frac{\varepsilon^2}{4\pi}\right)^2\int\Big[\frac{d}{dt}(j_\mu(\boldsymbol{r}_1)j_\lambda(\boldsymbol{r}_2))\,V_2(r),\,j_\nu(\boldsymbol{r}_1)j_\rho(\boldsymbol{r}_2)\,V_0(r)\Big]\,O_{\mu\rho}\,O_{\lambda\rho}\,d\boldsymbol{r}_1,\,d\boldsymbol{r}_2,\end{aligned} \tag{56}$$

$$V_0=\frac{e^{-\mu r}}{r},\qquad V_2=\frac{e^{-\mu r}}{2\mu}=-\frac{1}{2\mu}\,\frac{\partial}{\partial\mu}\,V_0. \tag{57}$$

(b) The non-static case. We make the approximation

$$j(1)\,j(3)\sim e^{2mi(t_1-t_3)} \tag{58}$$

since the main contribution comes from processes in which the initial and final state is positive, while the intermediate is negative, in energy. Here we must be careful about the sign of m in the exponential. It is determined by the order of the operators $j(1)$ and $j(3)$, but not by the order of the times t_1 and t_3. The integral corresponding to (48) is then

$$-\left\{\frac{1}{\left(2m-\frac{1}{2}\,(k_0-k_0')\right)^2}+\frac{1}{\left(2m+\frac{1}{2}\,(k_0-k_0')\right)^2}\right\}\,2\pi\delta(k_0+k_0')\int dT$$

for the order 13·42,

and
$$\left\{-\frac{2\pi i}{m}\,\delta(k_0-k_0')+\frac{2}{(2m)^2-\left(\frac{k_0-k_0'}{2}\right)^2}\right\}\,2\pi\delta(k_0+k_0')\int dT \tag{59}$$

for the order 13·24.

We neglect k_0 and k_0' compared to m, since the important contribution comes from values of k_0 of the order μ, which is $\ll m$ for actual cases. The interaction (A) is thus given by

$$H_A^{(4)}=\left(\frac{\varepsilon^2}{4\pi}\right)^2\frac{1}{2}\int\{j_\mu(1),O_{\mu\rho}j_\rho(2)\}^2\cdot\frac{1}{4m}\,V_0(r)^2\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2$$

$$+\left(\frac{\varepsilon^2}{4\pi}\right)^2\frac{1}{2}\int\{j_\mu(1),\,j_\nu(1)\}\{j_\lambda(2),\,j_\rho(2)\}O_{\mu\rho}\,O_{\nu\lambda}\,\frac{1}{8\pi}\,\frac{\mu}{m^2r^2}\,K_1(2\mu r)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{60}$$

The interaction (B), on the other hand, becomes zero in the present approximation. We shall not, however, enter into more detailed treatments.

c) Cross terms. When some j are static and others non-static, there occur in the expression (43) cross terms of these quantities. They may be calculated in a similar manner. But the resulting fourth order potentials will be non-static with radial dependence essentially the same as in the case b). Consequently their order of magnitude will be smaller than the preceding ones. We shall not investigate these terms here.

In the above results, a finite field mass μ has been assumed. If $\mu=0$, they cannot be applied at once since some of the expressions diverge. In this case we can show that

in (57) $$V_2\rightarrow\frac{r}{2},$$

and in (60) $$\frac{1}{8\pi}\,\frac{\mu}{m^2r^2}\,K_1(2\mu r)\rightarrow\frac{1}{16\pi}\,\frac{1}{m^2r^3}. \tag{61}$$

A reasonable evaluation of (50) for $\mu=0$ is not known, but it may be left out of consideration since fortunately such a case does not occur actually.

4. The Relation between the Second and Fourth Order Potential

Now we have at hand the second and fourth order potential $H^{(2)}$ and $H^{(4)}$. Their sum should be taken as the potential energy for the system which is rigorous up to the fourth order in the coupling contant. $H^{(2)}$ is a familiar interaction which can be derived by various methods, except that the retardation effect should be taken account of in a manner specified above. In case j is static, a retardation expansion is allowed. But for general j the situation is different. In fact, for transitions in which either of the particles changes its energy sign but not both, (odd transition), the interaction is virtually of a delta-function type which will not fit in the category of the ordinary potential. In other words, we had better decompose every quantity j into a static part $\bar{j}$ and a non-static (Zitterbewegung) part* $\tilde{j}$, and apply the retarded potential (38) only for the even part $\bar{j}_\lambda \bar{j}_\mu + \tilde{j}_\lambda \tilde{j}_\mu$. In this sense the Pauli approximation plays a major role in the two-body problem.

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* The terms static and non-static are somewhat deviating from ordinary usage. They refer to the change of the energy sign.

The fourth order potential, on the other hand, exhibits several interesting features. It takes on different forms according to the v/c-dependence of j as above. Moreover, it has an additional dependence on the spin variables through commutator expressions. Suppose that all the j's commute with each other, then the term (49) or the first term of (43) vanishes, leaving only the retardation correction (56) in the static case. Thus there remain no fourth order forces except for a small retardation correction.

Let us see how these fourth order characteristics show up in the electromagnetic interaction of two electrons. Here we have one static and three non-static interactions, namely

$$j_4(\boldsymbol{r}_1)\, j_4(\boldsymbol{r}_2) = -(\bar{\psi}\gamma_4\psi)_1\, (\bar{\psi}\gamma_4\psi)_2 \tag{62a}$$

and

$$j_i(\boldsymbol{r}_1)\, j_i(\boldsymbol{r}_2) = -(\bar{\psi}\gamma_i\psi)_1\, (\bar{\psi}\gamma_i\psi)_2, \quad i=1,2,3. \tag{62b}$$

We apply the formula (49) and (50) for (62a) and (60) for (62b). The former, however, vanishes, showing that the Coulomb potential is exact even to the fourth order effect. The latter yields a relatively large spin-spin and a smaller spin-independent interaction of the electrons, but the spin-spin term is just cancelled by a second order perturbation of the second order potential with $j_i \tilde{j}_i$ *via* negative state. The retardation correction (56) can be applied only to (62a). We can, however, again eliminate it by a canonical transformation (which is equivalent to the previously considered one (29)):

$$\Psi = \exp\left[\frac{i}{8}\,\frac{d}{dt}\int j_4(\boldsymbol{r}_1)\, V_2 j_4(\boldsymbol{r}_2)\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2\right]\Psi'. \tag{63}$$

In fact the transformed interaction becomes

$$\begin{aligned} &-\frac{1}{2}\int j_\lambda(1)\, j_\lambda(2)\, V - \frac{i}{8}\iint\left[\frac{d}{dt}\,(j_4(1)\, j_4(2)),\ j_4(1)\, j_4(2)\right] V_0 V_2 \\ &+\frac{i}{8}\iint\left[\frac{d}{dt}\,(j_4(1)\, j_i(2)),\ j_4(1)\, j_i(2)\right] V_0 V_2 + \frac{1}{8}\int \frac{d^2}{dt^2}\,(j_4(1)\, j_4(2))\, V_2 \end{aligned} \tag{64}$$

+ higher order terms.

The last term contributes

$$\frac{1}{8}\int\left\{\left(\frac{d}{dt_1}\right)^2 j_4(1)\, j_4(2) + 2\,\frac{d}{dt_1}\, j_4(1)\,\frac{d}{dt_2}\, j_4(2) + j_4(1)\left(\frac{d}{dt_2}\right)^2 j_4(2)\right\} V_2, \tag{65}$$

which combines with the first term with the expansion (38). Thus we get the corrected Breit interaction, with $j_\lambda j_\lambda$ replaced by $\bar{j}_\lambda \bar{j}_\lambda + \tilde{j}_\lambda \tilde{j}_\lambda \equiv \overline{j_\lambda j_\lambda}$,

$$H^{(2)\prime}=\frac{\varepsilon^2}{4\pi}\frac{1}{2}\int\left\{\overline{j_\lambda(1)\,j_\lambda(2)}\,V_0-\frac{1}{2}\frac{dj_4(1)}{dt}\frac{dj_4(2)}{dt}V_2\right\}d\boldsymbol{r}_1\,d\boldsymbol{r}_2$$

$$+\left(\frac{\varepsilon^2}{4\pi}\right)^2\frac{1}{2}\int\{\tilde{j}_i(1)\,\tilde{j}_i(2)\}^2\cdot\frac{1}{4m}V_0^2\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2+\text{smaller terms.}\qquad(66)$$

There remains no net fourth order effect of detectable magnitude. On the other hand, if we try to perform a second order perturbation using the original Breit formula, a result is obtained which contradicts the experimental facts about the hyperfine structure of helium[12]. This is due to the lack of the fourth order corrections which cancel a part of the iteration of the second order potential, leaving no spin-dependent fourth order effects. The same circumstance is observed in the (symmetrical or charged) pseudoscalar meson theory with pseudoscalar coupling investigated by Watson and Lepore[18], in which the isotopic spin variables replace the ordinary ones.

5. Further Discussions

In the preceding analysis we have given a general form of the potential which is correct up to the fourth order in the coupling constant and takes an approximate account of the retardation effect. The result shows that the corrections, which should be made to the potential directly derivable from non-relativistic classical theory, is essentially dependent on the nature of the interaction. Those interactions which contain only classical variables give in the quantum theory also potentials not much different from classical ones. On the other hand, if the interaction contains quantum variables, such as the Dirac spin and isotopic spin matrices, the higher order corrections in general affect the classical potential. Although these corrections have been considered only in the fourth approximation, we can in principle obtain the higher order ones step by step, in such a manner that the resulting potential yield exactly the *S*-matrix derived from the field theory. The origin of these correction lies, as was seen above, in the fact that the order of the emission and absorption of quanta exchanged between the particles affect the matrix elements. As an illustration, let us consider a charged meson theory, which contains the non-commutative isotopic spin variables τ_λ. The physical meaning of the non-commutativity is that the charge must be conserved at any instant, so that a nucleon cannot emit or absorb successively more than one positive or negative mesons. In other words, there is a strong correlation between the partaking quanta, which produces deviations in the higher order *S*-matrix terms from those obtained by a mere iteration of the second order process. For scattering problems these higher order corrections may be neglected. But for the bound system they would give in general a large influence on the eigenvalues and eigenfunctions. It is very difficult, however, to give an explicit expression for these corrections. We give below only an idea about these circumstances on a simple model, which may rather be too rough an approximation.

Take a symmetrical meson theory with static interaction. The main nth order processes consist of those graphs in which n meson lines connect the two nucleon lines, the order of the joints on the particle lines being different for each graph. Every graph gives rise to a matrix element of the form

$$\sum_{\tau}(\tau_1\tau_2\ldots\tau_n)_1\,(\tau_{i_1}\tau_{i_2}\ldots\tau_{i_n})_2\,f_n(p_1\,q_1\,p_2\,q_2,\ (i_1\ldots i_n)),\tag{67}$$

where p_1, q_1, p_2, q_2 are the initial and final momenta of the particles. Now we *assume* that the function f virtually does not depend on the permutation symbol $(i_1\ldots i_n)$ for the *irreducible* graphs, in which every meson line crosses some other meson lines.

For these graphs we put

$$f_n=\langle f_n\rangle_{\mathrm{Av}}\sim-\frac{1}{C_n}v_0^{\,n-1}v=-kv_0^{\,n},\tag{68}$$

$$v=ce^{-\mu r}/r,\qquad v_0\sim ce^{-\mu r}/\sqrt{\mu r},\quad k\sim\sqrt{\mu/r},$$

where C_n is the number of irreducible graphs. This form may be suggested by observing that

$$(-i)^2C_mf_m\cdot C_{n-m}f_{n-m}\cdot T\sim C_nf_n\tag{69}$$

and

$$\int\frac{e^{ikr}}{k^2+\mu^2}dk=4\pi K_0(\mu r)\sim\frac{1}{\sqrt{\mu r}}e^{-\mu r},\tag{70}$$

where $T\sim 1/k$ is the time interval in which a mth and an $(n-m)$th order graph overlap, giving an nth order irreducible graph. Then we extend for simplicity the average over the irreducible graphs to all graphs, thereby using the formula

$$\frac{1}{n!}\sum_{(i_1\ldots i_n)}(\tau_1\ldots\tau_n)_1\,(\tau_{i_1}\ldots\tau_{i_n})_2=\begin{cases}n+1 & n=\text{even},\\ \dfrac{n+2}{3}\,\tau_1\cdot\tau_2 & n=\text{odd}.\end{cases}\tag{71}$$

Next, starting with an effective potential

$$\{(\tau_1\cdot\tau_2)+a\}V=-k\sum s_nv_0^{\,n},\tag{72}$$

we get similar irreducible terms of S-matrix:

$$\{(\tau_1\cdot\tau_2)+a\}^nF_n,\quad F_n\sim-VV_0^{\,n-1}=-kV_0^{\,n}.\tag{73}$$

Equating both S-matrices, we obtain an equation determining the effective potential, which is solved as

$$1+\frac{1}{k}V=\frac{(A-2B)-B(\tau_1\cdot\tau_2)}{(A+B)(A-3B)},$$

$$A=\frac{1+v_0^2}{(1-v_0^2)^2},\quad B=\frac{1-v_0^2/3}{(1-v_0^2)^2}v_0. \tag{74}$$

This is a rather strange result. For large r and small coupling constant, V is nearly equal to $-v(\tau_1\cdot\tau_2)$. But for the contrary case v_0 will be large, and the isotopic dependence is of a linear combination of $\tau_1\cdot\tau_2$ and 1. Of course the above form of the functional dependence is not a reliable one, because of the nature of the present investigation. But it may be sufficient to make one modest in asserting the validity of lower order approximations.

Next we make a remark on the relation between the results obtained here and those which follows from the ordinary perturbation calculation. Writing the momentum-energy of the particle and emitted quantum as (p, p_0) and (k, k_0) respectively, the second order matrix element is

$$\left(\frac{1}{2\pi}\right)^3\left\{\frac{1}{-(\Delta p_0)_1-k_0}+\frac{1}{-(\Delta p_0)_2-k_0}\right\}O_1O_2\frac{1}{2k_0}. \tag{75}$$

The energy denominators can be rewritten as

$$\begin{aligned}\frac{1}{-(\Delta p_0)_1-k_0}+\frac{1}{-(\Delta p_0)_2-k_0}&=\frac{1}{-(\Delta p_0)_1-k_0}+\frac{1}{(\Delta p_0)_1-k_0}\\&+\frac{1}{-(\Delta p_0)_2-k_0}-\frac{1}{(\Delta p_0)_1-k_0}\\&=\frac{-2k_0}{k_0^2-(\Delta p_0)_1^2}+\frac{(\Delta p_0)_1+(\Delta p_0)_2}{\{k_0-(\Delta p)_1\}\{k_0+(\Delta p_0)_2\}}.\end{aligned} \tag{76}$$

The first term corresponds to the Eqs. (9) and (10), while the second is, being of the form $i(d/dt)V$ (non-Hermitic), contributes only to a higher order effect. The symmetrical form (37), however, does not follow from the perturbation theory. The fourth order interaction is therefore also different. In fact the perturbation theory will not give a fourth order potential as simple as the present one.

Thus far we have been confining ourselves to essentially non-relativistic treatments. Those regions where the distance of the particles is small and relativistic effects play a predominant role are out of consideration in the present investigation. Moreover, the difficulty of high singularities of the potential which haunts these regions is of a more profound origin and will only be solved with the general difficulties of the present quantum theory. A complete relativistic eigenvalue problem itself, if it could be formulated, would not give much help to these fundamental difficulties.

As for the relativistic formulation of the two-body problem, our results seem to present a slight progress over the older ones in that the many-time theory has been invoked as a guiding principle which requires a certain integrability condition for the interaction term. In view of the preceding analysis, we easily find that a pair of equations of the form (in the interaction representation)

$$\left\{i\frac{\partial}{\partial t_2}-V\left(\frac{t_1+t_2}{2}\right)\right\}\psi=0,\qquad \left\{i\frac{\partial}{\partial t_2}-V\left(\frac{t_1+t_2}{2}\right)\right\}\psi=0 \tag{77}$$

just meet our purpose. Transforming back to the Schrödinger representation, we have

$$\left\{i\frac{\partial}{\partial t_1}-H_1-V(t_1-t_2)\right\}\psi=0,\quad \left\{i\frac{\partial}{\partial t_2}-H_2-V(t_1-t_2)\right\}\psi=0, \tag{78}$$

or

$$\left\{i\frac{\partial}{\partial T}-(H_1+H_2)-2V(t)\right\}\psi=0,\qquad \left\{i\frac{\partial}{\partial t}-(H_1-H_2)\right\}\psi=0, \tag{79}$$

with

$$\begin{aligned} &i\frac{\partial V}{\partial t}-[H_1-H_2,\ V]=0, \qquad T=\frac{t_1+t_2}{2},\qquad t=t_1-t_2. \\ &V=e^{-i(H_1-H_2)t}\,V(0)\,e^{i(H_1-H_2)t}, \end{aligned} \tag{80}$$

The space-time behavior of the wave function ψ is such that it is essentially determined at $t_1=t_2$, and extended to $t_1\neq t_2$ simply as free of interaction. The latter procedure means nothing but a canonical transformation of the whole representation by a function $\exp[i(H_1-H_2)t]$. Thus the above many-time equations do not really present a purely relativistic (four-dimensional) formulation. They are essentially the same as the ordinary one-time equation. However, we notice here a fact which seems to have been overlooked. That is the freedom of canonical transformation just mentioned. Indeed any equation of the form

$$\left(i\frac{\partial}{\partial t}-H-V\right)\psi=0 \tag{81}$$

allows a transformation $\exp[if(H)]\psi$ with the result that only the interaction potential suffers a modification. In this sense the potential is not uniquely determined though the ensuing S-matrix is unique[14].

Finally a remark may be added concerning the Eqs. (31) and (32). The S-matrix (31) is unitary, but its sub-matrix under consideration is not. For if two particles collide, the Bremsstrahlang will occur, which diminishes the norm of the final elastic state. This means that V is in general complex. In fact an atom in an excited state will "decay" by emitting photons, and this should be properly represented by the complex potential. We may, however, regard V approximately Hermitic, at least at low energies. The preceding results give it some justification.

In conclusion, we may say that the concept of potential energy cannot enjoy a wide and practical extension much beyond the classical and non-relativistic form. Consequently a more ambitious attempt at a relativistic eigenvalue problem for closed systems will have to deviate considerably from the present one. An example will be furnished by the equation

$$\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} - \kappa\right)_1 \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - \kappa\right)_2 \psi = -\frac{\varepsilon^2}{2} (\gamma_\nu)_1 (\gamma_\nu)_2 D_F(12)\psi. \tag{82}$$

This equation can be deduced, as a reasonable one, from the many-time equation with intermediate field at the cost of some assumption and alteration on the meaning of state vector. We might expect that it determines an eigen-momentum-energy vector of the system from some boundary conditions at (space-time) infinity. Eq. (81) only takes account of the second order effect. Inclusion of higher order effects is not possible in the above form and can only be achieved by an integral equation. If higher order effects play an important part on the interaction, we shall therefore have to resort to still different methods, such as the S-matrix theory, the strong coupling theory, direct tackling with integral equations,[15] and so on.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On the Scattering of π-Mesons by Nucleons

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(Received July 31, 1950)

§ 1. Introduction.

The low cross section observed for the scattering of μ-mesons by nucleons was long a puzzle1) and was finally resolved after it was demonstrated that μ-mesons are very weakly coupled to nucleons. The strong interaction of π-mesons with nucleons, implied by their direct production in nucleon-nucleon collisions, raises the old question in a new form: should π-mesons suffer appreciable non-Coulombic scattering by nucleons? The present paper attempts to answer this question on the assumption that both the neutral and charged π-mesons possess spin zero.2) We confine ourselves to spin zero charged π's because at least for the scalar field with scalar coupling and the pseudoscalar field with pseudoscalar coupling, we can obtain the higher-order corrections to the scattering cross sections by the usual methods of charge and mass renormalization. This is not true for spin one charged meson fields. We believe that some interest attaches to a comparison with experiment of those predictions of weak coupling meson theories for which the higher-order corrections can be calculated.

In § 2 we derive the lowest order expressions (valid for all meson energies) for the differential and total scattering cross sections predicted by the scalar and pseudoscalar weak coupling theories. Charge-exchange scattering as well as direct scattering is considered.3) For the sake of completeness and also because they are so qualitatively different, the cross sections are also recorded in lowest order for the derivative couplings, although the latter can not be handled by the customary renormalization procedures (i. e. scalar theory with vector coupling, pseudoscalar theory with pseudovector coupling). Some representative numerical

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1) The old measurements (cf. R. P. Shutt, Phys. Rev. **61** (1942), 6 and references cited therein) led to an upper limit of 10^{-28} cm^2; recently, Amaldi and Fidecara (Helv. Phys. Acta **23** (1950), 93) pushed the upper limit down to $5\cdot10^{-30}$ cm^2, a value approaching the Coulomb cross section for single large angle scattering.

2) The evidence is conclusive (cf. J. Steinberger, W. Panofsky and J. Steller, Phys. Rev. **78** (1950), 802) that the spin of the neutral π-meson is zero however, it is still not certain that the spin of the charged π-meson is zero.

3) These lowest order cross sections have also been derived by M. Peshkin in a paper soon to appear in the Physical Review. We wish to thank Mr. Peshkin for a comparison of results.

results are tabulated for both the total cross sections and the angular distributions (cf. Tables I, II, III). The work is reported in considerable detail in the interest of clarity and as an illustration of the power of the new perturbation techniques. In § 3, the higher-order corrections to the direct and charge-exchange scattering are calculated for the non-derivative couplings for spin zero meson using Feynman's method;[4] we have also checked our results with Dyson's method.[5] We apply the general formulae to obtain numerical answers in the non-relativistic (Thomson) limit.[6] Finally, in § 4, we compare the theoretical predictions with existing experiments.

§ 2. Scattering Calculations in the Lowest Order.

(a) *Matrix Elements for Scattering Processes.*

Perturbation theoretic calculations of matrix elements for meson-nucleon interactions are greatly simplified by the relativistically covariant perturbation methods introduced by Feynman[4] and we shall use these methods throughout for the discussion of the meson-nucleon scattering. Restricting the considerations to the scattering of charged mesons (π^+ and π^-) by neutrons or protons (N or P), and allowing neutral mesons (π^0) among the final products, there are six possible scattering reactions, conveniently separated into three categories:

$$\begin{aligned}
&\text{I}: && \pi^- + P \to \pi^- + P \quad \text{or} \quad \pi^+ + N \to \pi^+ + N \\
&\text{II}: && \pi^- + N \to \pi^- + N \quad \text{or} \quad \pi^+ + P \to \pi^+ + P \\
&\text{III}: && \text{Charge exchange scattering} \\
& && \pi^- + P \to \pi^0 + N \quad \text{or} \quad \pi^+ + N \to \pi^0 + P.
\end{aligned} \tag{1}$$

The lowest order Feynman diagrams for the three types of scattering are shown in Fig. 1. In the diagram corresponding to I the incident meson is absorbed by the nucleon prior to the emission of the final meson since a proton can absorb but not emit π^- and a neutron can absorb but not emit π^+. For process II the emission of the final meson precedes the absorption of the incident meson. Both diagrams evidently contribute to the charge exchange scattering, III. With each diagram goes a relativistically invariant matrix element which may be immediately

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4) R. P. Feynman, Phys. Rev. **76** (1949), 748 and **76** (1949), 760.

5) F. J. Dyson, Phys. Rev. **75** (1949), 486, and **75** (1949), 1736; we are indebted to Mr. Dyson for a lucid presentation of his method.

6) A recent paper by E. Corinaldesi and G. Field (Phil. Mag. **40** (1949), 1159) contains the correction for the process $\pi^+ + N \to \pi^+ + N$ in the Thomson limit for pseudoscalar mesons with pseudoscalar coupling using the older techniques. This result agrees with ours in the limit of vanishing mass for the meson. A further paper by Corinaldesi and Field (Phil. Mag. **41** (1950), 364) extends their considerations to finite kinetic energies of the meson, small compared with the nucleon rest energy.

written down according to the different theories chosen to describe the mesons (the nucleons are assumed to obey the Dirac equation). For mesons of spin zero there are four possibilities which we list together with the factor in the Feynman matrix element to be inserted upon absorption or emission of a meson (with energy-momentum four-vector k):

The scattering cross sections for processes I, II, and III have been worked out for each of the theories (2).

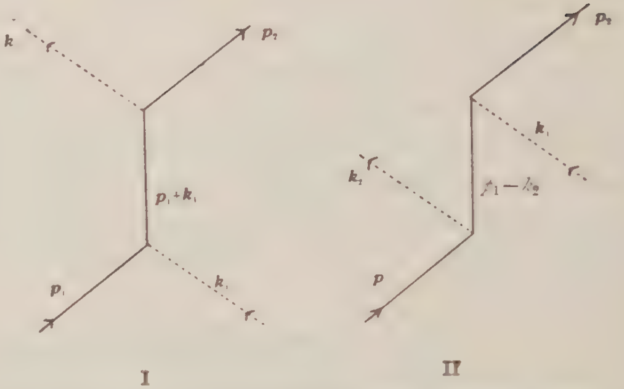

Fig. 1. Feynman diagrams for scattering processes I and II. For the charge exchange scattering, III, the diagrams I and II contribute interfering amplitudes. Solid lines indicate the nucleons and dotted lines indicate the mesons.

	Type of Meson	Type of Coupling	Factor for meson k	
			Absorption	Emission
$PS(PS)$	Pseudoscalar	Pseudoscalar	$g\gamma_5$	$g\gamma_5$
$PS(PV)$	Pseudoscalar	Pseudovector	$(g/\mu)\gamma_5\boldsymbol{k}$	$-(g/\mu)\gamma_5\boldsymbol{k}$
$S(S)$	Scalar	Scalar	g	g
$S(V)$	Scalar	Vector	$(g/\mu)\boldsymbol{k}$	$-(g/\mu)\boldsymbol{k}$

(2)

The notation in (2) and in what follows will be identical with that of Feynman: Units will be chosen so that $\hbar=c=1$. M will denote the nucleon mass, μ the meson mass, and we shall not differentiate with respect to mass between neutrons and protons or between neutral and charged mesons.[7] The dimensionless "mesic charge" g (really $g/(\hbar c)^{1/2}$) measures the meson-nucleon interaction and is different for the different theories. The scalar product of two four-vectors will be written in the form $A \cdot B = A_\mu B_\mu = A^4B^4 - A_1B_1 - A_2B_2 - A_3B_3$, and with this summation convention we may associate to each four-vector p a Dirac operator $\boldsymbol{p}=p_\mu\gamma_\mu$ where the γ_μ are the four matrices $\beta\vec{\alpha}$, β. The product of two operators $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfies the relation $\boldsymbol{AB}+\boldsymbol{BA}=2A\cdot B$. A nucleon of energy-momentum p is characterized by a spinor u obeying the Dirac equation $\boldsymbol{p}u=Mu$ or $\bar{u}\boldsymbol{p}=M\bar{u}$, where $\bar{u}$ is related to the hermitian conjugate u^* by $\bar{u}=u^*\beta$.

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[7] For the charge exchange reaction the mass differences are, of course important, especially for very small meson kinetic energy. Since the π^0: π^+ mass difference is sufficient, under these conditions, to allow the charge exchange reaction to take place (cf. ref. 2)) but is otherwise only fairly well known, we prefer to neglect this difference and the N, P mass difference.

The matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ which appears in the pseudoscalar theories is anti-hermitian ($\gamma_5^* = -\gamma_5$, $\gamma_5^2 = -1$), and anti-commutes with the four matrices γ_μ.

Let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ denote the momenta of the initial and final nucleons, and $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ the momenta of the initial and final mesons. ($\boldsymbol{p}_1 + \boldsymbol{k}_1 = \boldsymbol{p}_2 + \boldsymbol{k}_2$ by energy-momentum conservation; $\boldsymbol{p}_1^2 = \boldsymbol{p}_2^2 = M^2$ and $\boldsymbol{k}_1^2 = \boldsymbol{k}_2^2 = \mu^2$). According to Feynman's rules, the invariant matrix-elements for the scattering process I, for each of the theories (2), are:[8]

$$\begin{array}{ll} PS(PS) & g^2\mathfrak{M}_I = -ig^2\bar{u}_2\gamma_5(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1}\gamma_5 u_1, \\ PS(PV) & g^2\mathfrak{M}_I = -i(g^2/\mu^2)\bar{u}_2(-\gamma_5\boldsymbol{k}_2)(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1}\gamma_5\boldsymbol{k}_1 u_1, \\ S(S) & g^2\mathfrak{M}_I = -ig^2\bar{u}_2(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1}u_1, \\ S(V) & g^2\mathfrak{M}_I = -i(g^2/\mu^2)\bar{u}_2(-\boldsymbol{k}_2)(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1}\boldsymbol{k}_1 u_1. \end{array} \tag{3}$$

Between the absorption of the meson $\boldsymbol{k}_1$ and emission of the meson $\boldsymbol{k}_2$ (represented by the factors given in (2)), the nucleon propagates in a virtual state with momentum $\boldsymbol{p}_1 + \boldsymbol{k}_1$ and propagation factor $(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1}$. In (3), u_1 and u_2 represent the spinors appropriate to the initial and final states of the nucleon: $\boldsymbol{p}_1 u_1 = M u_1$ and $\bar{u}_2\boldsymbol{p}_2 = M\bar{u}_2$. The normalization is chosen so that $(\bar{u}u) = 1$.

The matrix elements (3) can be considerably simplified by making use of the Dirac equations for the initial and final nucleon states and of the properties of γ_5 where this operator occurs. We illustrate the procedure with the most complicated of the four cases, $PS(PV)$. Writing the propagation factor in the form $(\boldsymbol{p}_1 + \boldsymbol{k}_1 - M)^{-1} = (\boldsymbol{p}_1 + \boldsymbol{k}_1 + M)(\mu^2 + 2p_1 \cdot k_1)^{-1}$ and using the fact that any $\boldsymbol{p}_1$ occurring to the extreme right or $\boldsymbol{p}_2$ occurring to the extreme left may be replaced by M in the matrix element, we have

$$\begin{aligned} \gamma_5\boldsymbol{k}_2(\boldsymbol{p}_1 + \boldsymbol{k}_1 + M)\gamma_5\boldsymbol{k}_1 &= \boldsymbol{k}_2(-\boldsymbol{p}_1 - \boldsymbol{k}_1 + M)\boldsymbol{k}_1 = \boldsymbol{k}_2(-\boldsymbol{p}_1\boldsymbol{k}_1 - \mu^2 + M\boldsymbol{k}_1) \\ &= \boldsymbol{k}_2(\boldsymbol{k}_1\boldsymbol{p}_1 - 2p_1 \cdot k_1) - \mu^2\boldsymbol{k}_2 + M\boldsymbol{k}_2\boldsymbol{k}_1 = 2M\boldsymbol{k}_2\boldsymbol{k}_1 - (\mu^2 + 2p_1 \cdot k_1)\boldsymbol{k}_2. \end{aligned}$$

The operator $\boldsymbol{k}_2\boldsymbol{k}_1$ which is bilinear in the γ-matrices may be reduced to an expression linear in the γ's and thus put it in a form more convenient for the eventual squaring of the matrix element. Now

$$(\boldsymbol{p}_2 + \boldsymbol{k}_2)(\boldsymbol{p}_1 + \boldsymbol{k}_1) = (\boldsymbol{p}_1 + \boldsymbol{k}_1)^2 = M^2 + \mu^2 + 2p_1 \cdot k_1.$$

But

$$\begin{aligned} (\boldsymbol{p}_2 + \boldsymbol{k}_2)(\boldsymbol{p}_1 + \boldsymbol{k}_1) &= \boldsymbol{p}_2\boldsymbol{p}_1 + \boldsymbol{k}_2\boldsymbol{p}_1 + \boldsymbol{p}_2\boldsymbol{k}_1 + \boldsymbol{k}_2\boldsymbol{k}_1 \\ &= M^2 + M(\boldsymbol{k}_1 + \boldsymbol{k}_2) + \boldsymbol{k}_2\boldsymbol{k}_1 \end{aligned}$$

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[8] The factor $-i$ in front of the matrix elements is obtained as follows. Each interaction with an external meson contributes a factor $-i$ from the perturbation theory, and there is another i from the Fourier transform, $i(2\pi)^{-2}(p - M)^{-1}$, of Feynman's propagation function $K_+(2, 1)$ for nucleons. The $4\pi^2$ is removed by the integrations performed in the transition from configuration to momentum space. A common factor $(2\pi)^4\delta(p_1 + k_1 - p_2 - k_2)$ obtained as a result of this integration is not written.

taken between $\bar{u}_2$ and u_1. Furthermore from the fact that $\bar{u}_2(\boldsymbol{p}_1+\boldsymbol{k}_1)u_1=\bar{u}_2(\boldsymbol{p}_2+\boldsymbol{k}_2)u_1$ we conclude that $\bar{u}_2\boldsymbol{k}_1u_1=u_2\boldsymbol{k}_2u_1$. Therefore, in the matrix element we may make the replacement

$$\boldsymbol{k}_2\boldsymbol{k}_1=\mu^2+2p_1\cdot k_1-2M\boldsymbol{k}_1.$$

All together we have, in the matrix element, for the $PS(PV)$ case,

$$\gamma_5\boldsymbol{k}_2(\boldsymbol{p}_1+\boldsymbol{k}_1+M)\gamma_5\boldsymbol{k}_1=2M(\mu^2+2p_1\cdot k_1)-[4M^2+\mu^2+2p_1\cdot k_1]\boldsymbol{k}_1.$$

With the help of such devices we find for the various matrix elements

$$\begin{aligned}
&PS(PS) && g^2\mathfrak{M}_I=-ig^2(\mu^2+2p_1\cdot k_1)^{-1}(\bar{u}_2\boldsymbol{k}_1u_1),\\
&PS(PV) && g^2\mathfrak{M}_I=+i(g^2\ \ \mu^2)\bar{u}_2(2M-R_1\boldsymbol{k}_1)u_1,\\
&S(S) && g^2\mathfrak{M}_I=-ig^2(\mu^2+2p_1\cdot k_1)^{-1}\bar{u}_2(2M+\boldsymbol{k}_1)u_1,\\
&S(V) && g^2\mathfrak{M}_I=+i(g^2/\mu^2)\bar{u}_2\boldsymbol{k}_1u_1
\end{aligned}\tag{4}$$

where the quantity R_1 for $PS(PV)$ is defined in (6).

For scattering of type II we obtain

$$\begin{aligned}
&PS(PS) && g^2\mathfrak{M}_{II}=-ig^2(2p_1\cdot k_2-\mu^2)^{-1}(\bar{u}_2\boldsymbol{k}_1u_1),\\
&PS(PV) && g^2\mathfrak{M}_{II}=+i(g^2/\mu^2)\bar{u}_2(2M-R_2\boldsymbol{k}_1)u_1,\\
&S(S) && g^2\mathfrak{M}_{II}=+ig^2(2p_1\cdot k_2-\mu^2)^{-1}\bar{u}_2(2M-\boldsymbol{k}_1)u_1,\\
&S(V) && g^2\mathfrak{M}_{II}=-i(g^2/\mu^2)\bar{u}_2\boldsymbol{k}_1u_1.
\end{aligned}\tag{5}$$

The quantities R_1 and R_2 in the $PS(PV)$ case are

$$R_1=\frac{4M^2+\mu^2+2p_1\cdot k_1}{2p_1\cdot k_1+\mu^2},\ R_2=\frac{4M^2+\mu^2-2p_1\cdot k_2}{2p_1\cdot k_2-\mu^2}.\tag{6}$$

In the charge exchange scattering III we get contributions from diagram I and II, the final neutral meson being emitted in one case by a proton and in the other by a neutron. Let us denote by g_P and g_N the mesic charge of proton and neutron, respectively, for the coupling with the neutral meson field. Using values of g_P and g_N appropriate to the different theories (2), the matrix elements may be written

$$\begin{aligned}
&\pi^-+P\rightarrow\pi^0+N && \mathfrak{M}_{III}=g\{g_N\mathfrak{M}_I+g_P\mathfrak{M}_{II}\},\\
&\pi^++N\rightarrow\pi^0+P && \mathfrak{M}_{III}=g\{g_P\mathfrak{M}_I+g_N\mathfrak{M}_{II}\}
\end{aligned}\tag{7}$$

where $\mathfrak{M}_I$ and $\mathfrak{M}_{II}$ are taken from (4) and (5). There is of course no reason to assume, as we have done, that the charged and neutral meson fields have the same parity and the same type of coupling with the nucleons but we restrict ourselves to this case.

To obtain the cross sections for the processes I, II, and III we need the absolute squares of the matrix elements in (4), (5), and (7) averaged over the spin of the initial nucleon and summed over the spin of the final nucleon. The spin sums are performed with the help of Feynman's projection operators $(2M)^{-1}(\boldsymbol{p}_1+M)$ and $(2M)^{-1}(\boldsymbol{p}_2+M)$ applied to the spinors u_1 and u_2 respectively. In taking the absolute squares of (4), (5) and (7), four distinct quantities appear and give rise to the following spurs when summed and averaged over the spins :

$$\begin{aligned}
&\text{i)} && M^{-2}(\bar{u}_2\boldsymbol{k}_1u_1)(\bar{u}_1\boldsymbol{k}_1u_2) &&\rightarrow (8M^4)^{-1}Sp[\boldsymbol{k}_1(\boldsymbol{p}_1+M)\boldsymbol{k}_1(\boldsymbol{p}_2+M)],\\
&\text{ii)} && 4(\bar{u}_2u_1)(\bar{u}_1u_2) &&\rightarrow (2M^2)^{-1}Sp[(\boldsymbol{p}_1+M)(\boldsymbol{p}_2+M)\ ,\\
&\text{iii)} && 4M^{-1}(\bar{u}_2\boldsymbol{k}_1u_1)(\bar{u}_1u_2) &&\rightarrow (2M^3)^{-1}Sp[\boldsymbol{k}_1(\boldsymbol{p}_1+M)(\boldsymbol{p}_2+M)],\\
&\text{iv)} && 4M^{-1}(\bar{u}_2u_1)(\bar{u}_1\boldsymbol{k}_1u_2) &&\rightarrow (2M^3)^{-1}Sp[(\boldsymbol{p}_1+M)\boldsymbol{k}_1(\boldsymbol{p}_2+M)].
\end{aligned}$$

Evaluation of the spurs shows that iii) and iv) are identical. The absolute squares of the matrix elements are therefore expressible in terms of three quantities A, B, and C obtained from the spurs i), ii) and iii) :

$$\begin{aligned}
&\text{i)} && A\equiv(2M^4)^{-1}\{2(k_1\cdot p_1)(k_1\cdot p_2)-\mu^2(p_1\cdot p_2-M^2)\},\\
&\text{ii)} && B\equiv(2/M^2)\{p_1\cdot p_2+M^2\},\\
&\text{iii)} && C\equiv(2/M^2)\{k_1\cdot p_1+k_1\cdot p_2\}.
\end{aligned}\tag{8}$$

The squares of the matrix elements (4) for process I are

$$\begin{aligned}
&PS(PS) && |H|_I^2=g^4M^2(2p_1\cdot k_1+\mu^2)^{-2}A,\\
&PS(PV) && |H|_I^2=g^4M^2\mu^{-4}\{R_1^2A+B-R_1C\},\\
&S(S) && |H|_I^2=g^4M^2(2p_1\cdot k_1+\mu^2)^{-2}\{A+B+C\},\\
&S(V) && |H|_I^2=g^4M^2\mu^{-4}A.
\end{aligned}\tag{9}$$

For process II we obtain

$$\begin{aligned}
&PS(PS) && |H|_{II}^2=g^4M^2(2p_1\cdot k_2-\mu^2)^{-2}A,\\
&PS(PV) && |H|_{II}^2=g^4M^2\mu^{-4}\{R_2^2A+B-R_2C\},\\
&S(S) && |H|_{II}^2=g^4M^2(2p_1\cdot k_2-\mu^2)^{-2}\{A+B-C\},\\
&S(V) && |H|_{II}^2=g^4M^2\mu^{-4}A.
\end{aligned}\tag{10}$$

The squares of the charge exchange matrix elements, for the reaction $\pi^-+P\rightarrow\pi^0+N$, are :

$$PS(PS)\quad |H|_{III}^2=g^2M^2A\left\{\frac{g_N}{2p_1\cdot k_1+\mu^2}+\frac{g_P}{2p_1\cdot k_2-\mu^2}\right\}^2,$$

$$\begin{aligned}
PS(PV)\quad |H|_{III}^2=g^2M^2\mu^{-4}\{&A(R_1g_N+R_2g_P)^2+B(g_N+g_P)^2\\
&-C(R_1g_N^2+(R_1+R_2)g_Ng_P+R_2g_P^2)\},
\end{aligned}$$

$$S(S) \qquad |H|^2_{III} = g^2M^2A\left\{\frac{g_N}{2p_1\cdot k_1+\mu^2} + \frac{g_P}{2p_1\cdot k_2-\mu^2}\right\}^2$$

$$+ g^2M^2B\left\{\frac{g_N}{2p_1\cdot k_1+\mu^2} - \frac{g_P}{2p_1\cdot k_2-\mu^2}\right\}^2$$

$$+ g^2M^2C\left\{\frac{g_N^2}{(2p_1\cdot k_1+\mu^2)^2} - \frac{g_P^2}{(2p_1\cdot k_2-\mu^2)^2}\right\},$$

$$S(V) \qquad |H|^2_{III} = g^2M^2\mu^{-4}(g_N-g_P)^2A. \tag{11}$$

For the other reaction $\pi^+ + N \rightarrow \pi^0 + P$ we must interchange g_N and g_P in (11).

(b) *Differential Scattering Cross Sections in the Laboratory System.*

For comparison with eventual experiments it is convenient to express the differential cross section in the laboratory frame of reference in which the target nucleon is initially at rest. To be sure, the angular distribution of the scattering in the center of mass ($C.M.$) coordinate system (total momentum equal to zero) gives a more vivid reflection of the different assumptions for the meson-nucleon interaction ; thus if the square of the matrix element, $|H|^2$, is independent of the angle of scattering, there is an isotropic distribution in the $C.M.$ system, and deviations from this are due to the angular dependence of $|H|^2$. In the laboratory coordinate system, however, a certain part of the angular dependence of the cross section is of purely kinematic origin and expresses the ratio of the elements of solid angle for a scattered particle in transforming from the $C.M.$ to the laboratory system. This part of the angular dependence is easily separated out so that there is no difficulty in locating the effects of the different meson theories.

Let us denote by $k_1 = (E_1, \vec{K}_1)$ and $k_2 = (E_2, \vec{K}_2)$ the energy-momentum four-vectors of the incident and scattered mesons, respectively, in the laboratory system. ($E_1^2 = K_1^2 + \mu^2$ and $E_2^2 = K_2^2 + \mu^2$). Similarly let $p_1 = (M, 0)$ and $p_2 = (W_2, \vec{P}_2)$ represent the energy-momentum four-vectors for the initial and final nucleon states. ($W_2^2 = P_2^2 + M^2$). Energy and momentum conservation give $E_1 + M = E_2 + W_2$ and $\vec{K}_1 = \vec{K}_2 + \vec{P}_2$. The differential cross section for the scattering is given in terms of the square of the matrix element, $|H|^2$, and the density of final states, ρ_F, by

$$d\sigma = 2\pi\frac{\rho_F}{\text{current}}|H|^2\frac{M}{M}\cdot\frac{M}{W_2}\cdot\frac{2\pi}{E_1}\cdot\frac{2\pi}{E_2}.$$

The incident current, with the normalization chosen, is simply the velocity of the incident meson, K_1/E_1. The factors M/M and M/W_2 arise in correcting the normalization of the spinors u_1 and u_2 from $(\bar{u}u) = 1$ to $(u^*u) = 1$.[9] Normalization of the meson wave functions brings in the factors $2\pi/E_1$ and $2\pi/E_2$.

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[9] R. P. Feynman, reference 4, page 758.

In computing the density of final states we need only consider the volume in phase space associated with *one* of the final particles since the momentum of the other is immediately fixed by the conservation laws. Fixing our attention on the final meson we have for the density of final states, ρ_F,

$$\rho_F dE_F = (2\pi)^{-3} K_2^2 dK_2 d(\cos\theta)\, d\phi$$

where dE_F is a differential increment in the total energy of the final state and θ, ϕ measure the polar and azimuthal angles, respectively, of $\vec{K}_2$ with respect to $\vec{K}_1$ as axis. An especially simple form of the cross section results if we choose to express the density of states, ρ_F, as differential in K_2 and ϕ, regarding the differential increment in final energy, dE_F as arising from an increment $d(\cos\theta)$ keeping K_2 and ϕ fixed. With $E_F = E_2 + W_2$ and $W_2 = (M^2 + K_2^2 + K_1^2 - 2K_1K_2 \cos\theta)^{1/2}$, we have $(\partial E_F / \partial \cos\theta) = K_1K_2/W_2$. The differential cross section is thus

$$d\sigma = (2\pi)^3 |H|^2 \frac{E_1}{K_1} \cdot \frac{M}{W_2 E_1 E_2} \cdot \frac{1}{(2\pi)^3} \cdot \frac{W_2}{K_1 K_2} K_2^2 dK_2 d\phi$$

$$= |H|^2 \frac{MK_2}{K_1^2 E_2} dK_2 d\phi.$$

Since $E_2 dE_2 = K_2 dK_2$, we obtain finally

$$d\sigma = (M/K_1^2)\, |H|^2 d\phi dE_2. \tag{12}$$

The form (12) is very convenient for the subsequent integration to obtain the total cross section since the $|H|^2$ in (9), (10) and (11) are all expressible as functions of E_2 alone. For the angular distribution of the scattering, however, we must express E_2 as a function of the meson scattering angle θ. The most revealing way to do this is to consider the scattering first in the *C.M.* system and then to make a Lorentz transformation to the laboratory system. If θ^* is the angle of scattering of the meson, and γ^*, Γ^* refer respectively to the energies of the meson and nucleon in units of their respective rest masses, all quantities in the *C.M.* system, we find:

$$\begin{aligned} &\text{i)} \quad E_2 = \mu\gamma^*\Gamma^* + M(\Gamma^{*2} - 1)\cos\theta^* \\ &\text{ii)} \quad \mu\sqrt{\gamma^{*2} - 1} = M\sqrt{\Gamma^{*2} - 1} \\ &\text{iii)} \quad \Gamma^* = \frac{\gamma + M/\mu}{\sqrt{1 + 2(M/\mu)\gamma + (M/\mu)^2}} \end{aligned} \tag{13}$$

where γ is the energy of the incident meson in the laboratory system, in units of its rest mass; $\gamma = E_1/\mu$. The expression of $\cos\theta^*$ in terms of the laboratory quantity $\cos\theta$ gives the results

$$\text{i)} \quad \cos\theta^* = \frac{(1 + 2(M/\mu)\gamma + (M/\mu)^2)\cos\theta\sqrt{1 - (\mu/M)^2\sin^2\theta} - (\gamma + \mu/M)(\gamma + M/\mu)\sin^2\theta}{(\gamma + M/\mu)^2 - (\gamma^2 - 1)\cos^2\theta},$$

$$\text{ii)}\quad E_2=M\frac{(\gamma+\mu/M)(\gamma+M/\mu)+(\gamma^2-1)\cos\theta\sqrt{1-(\mu/M)^2\sin^2\theta}}{(\gamma+M/\mu)^2-(\gamma^2-1)\cos^2\theta}, \tag{14}$$

$$\text{iii)}\quad J=\frac{d(\cos\theta^*)}{d(\cos\theta)}=\frac{(1+2(M/\mu)\gamma+(M/\mu)^2)\left\{\dfrac{(\gamma+\mu/M)\cos\theta}{\sqrt{1-(\mu/M)^2\sin^2\theta}}+(\gamma+M/\mu)\sqrt{1-(\mu/M)^2\sin^2\theta}\right\}^2}{\{(\gamma+M/\mu)^2-(\gamma^2-1)\cos^2\theta\}^2}.$$

Since dE_2, according to (13), is equal to $d(\cos\theta^*)$ multiplied by a quantity independent of the scattering angle, we see that the purely kinematic factor in the laboratory angular distribution, $dE_2/d\cos\theta$, is simply proportional to the ratio of solid angles $J=d(\cos\theta^*)/d(\cos\theta)$ given by (14, iii). In the tables of numerical results given in part (d) of this section we have given values of J for meson scattering angles $\theta=0°$, $90°$, and $180°$ for incident meson energies $\gamma=1$, 3 and 10. The discussion of the angular distributions given by (9), (10), (11) in conjunction with (12), (13) and (14) is postponed to § 2(d).

Evaluation of the quantities A, B, and C of (8) and hence the squared matrix elements (9), (10), (11) in the laboratory system is facilitated by noting that all of the products of four-vectors in (8) may be made to contain $p_1=(M, 0)$ as a factor. Thus, from $p_1-k_2=p_2-k_1$ we conclude, by squaring both sides, that $k_1\cdot p_2=k_2\cdot p_1=ME_2$. Note also that $p_1\cdot p_2=MW_2=M^2+M(E_1-E_2)$. The expressions for A, B, C of (8) and R_1, R_2 of (6) are therefore

$$\begin{aligned}
&\text{i)}\quad A=E_1E_2/M^2-(\mu^2/2M^3)(E_1-E_2),\\
&\text{ii)}\quad B=4+(2/M)(E_1-E_2),\\
&\text{iii)}\quad C=(2/M)(E_1+E_2),\\
&\text{iv)}\quad R_1=\frac{4M^2+\mu^2+2ME_1}{2ME_1+\mu^2},\\
&\text{v)}\quad R_2=\frac{4M^2+\mu^2-2ME_2}{2ME_2-\mu^2}.
\end{aligned} \tag{15}$$

For experimental purposes it is useful to have the correlation between the scattering angle, θ, for the meson and the angle, ψ, between the direction of the recoiling nucleon and that of the incident meson. In terms of the $C.M.$ angle θ^* we readily find

$$\tan\psi=(1/\Gamma^*)\cot 1/2\,\theta^* \tag{16}$$

where Γ^* may be obtained from (13, iii) and θ^* expressed in terms of θ by (14, i). The kinetic energy of the recoiling nucleon coming off at the angle ψ is

$$W_2-M=\frac{2M(\Gamma^{*2}-1)}{1+\Gamma^{*2}\tan^2\psi}. \tag{17}$$

(c) *Total Scattering Cross Sections*

It is a simple matter to obtain the total cross sections from (12), (9), (10), (11) and (15) by performing the integration of the differential cross section with respect to E_2. (Integration of the azimuthal angle gives a factor 2π). The limits of integration for E_2 are obtained from (14) by setting θ equal to 0° for the maximum E_2(equal to E_1) and θ equal to 180° for the minimum E_2.

$$Min\ (E_2) = E_2(180°) = \frac{(M^2+\mu^2)E_1 + 2M\mu^2}{M^2+\mu^2+2ME_1}. \tag{18}$$

We give the results of the integration for each of the different scattering processes (1) and meson theories (2). We also give the expression in each case for the cross section in the extreme non-relativistic limit ($N.R.$) of zero kinetic energy for the meson (designated as the Thomson limit by analogy with the electromagnetic case) and in the extreme relativistic limit ($E.R.$).

For process I the cross sections are:

$$PS(PS) \quad \sigma_I = \frac{2\pi M^2 g^4\{2ME_1^3 + (2M^2+\mu^2)E_1^2 + 3M\mu^2 E_1 + \mu^4\}}{(2ME_1+\mu^2)^2(2ME_1+\mu^2+M^2)^2},$$

$$\sigma_I(N.R.) = \pi(g^2/M)^2[1+(\mu/M)]^{-2}(1+\mu/M)^{-2}; \sigma_I(E.R.) = \pi g^4(4ME_1)^{-1}, \tag{19}$$

$PS(PV)$

$$\sigma_I = 4\pi M^2 g^4 \mu^{-4}(2ME_1+\mu^2)^{-2}(2ME_1+\mu^2+M^2)^{-2}\{4M^3(E_1+M)(E_1^2-\mu^2)^2 +$$
$$+\mu^2(E_1^2-\mu^2)(6M^2E_1^2+3M\mu^2E_1--3M^2\mu^2+\tfrac{1}{2}\mu^4)+2M\mu^6(E_1-\mu)+\mu^6(M+\mu)^2\}, \tag{20}$$

$$\sigma_I(N.R.) = \text{Same as } PS(PS) \text{ (in units of } g^2); \quad \sigma_I(E.R.) = \pi(g/\mu)^4 ME_1,$$

$$S(S) \quad \sigma_I = \frac{2\pi M^2 g^4\{2ME_1^3 + (18M^2+\mu^2)E_1^2 + (24M^3+10M\mu^2)E_1 + 8M^4 + 8M^2\mu^2 + \mu^4\}}{(2ME_1+\mu^2)^2(2ME_1+\mu^2+M^2)^2}, \tag{21}$$

$$\sigma_I(N.R.) = 4\pi(g^2/\mu)^2(1+\mu/M)^{-2}; \quad \sigma_I(E.R.) = \pi g^4(4ME_1)^{-1},$$

$$S(V) \quad \sigma_I = \frac{2\pi M^2 g^4\{2ME_1^3 + (2M^2+\mu^2)E_1^2 + 2M\mu^2 E_1 + \mu^4\}}{\mu^4(2ME_1+\mu^2+M^2)^2}, \tag{22}$$

$$\sigma_I(N.R.) = \text{Same as } S(S) \text{ (in units of } g^2); \quad \sigma_I(E.R.) = \pi(g/\mu)^4 ME_1.$$

For process II there is a characteristic logarithmic factor entering which we shall abbreviate as "log":

$$\log \equiv \log\left\{1 + \frac{4M^2(E_1^2-\mu^2)}{2M^3E_1+3M^2\mu^2-\mu^4}\right\}. \tag{23}$$

The cross section for process II are:

$$PS(PS) \quad \sigma_{II} = \pi\left(\frac{g^2}{M}\right)^2\left\{\frac{2ME_1+\mu^2}{4(E_1^2-\mu^2)}\log + \frac{M^2\mu^4}{(4M^3E_1-2\mu^4)(ME_1+\mu^2)-3\mu^4(M^2-\mu^2)}\right\},$$

$$\sigma_{II}(N.R.) = \pi(g^2/M)^2[1-(\mu/2M)]^{-2}(1+\mu/M)^{-2}; \tag{24}$$

$$\sigma_{II}(E.R.) = \pi g^4(2ME_1)^{-1}\log(2E_1M^{-1}),$$

PS(PV)

$$\sigma_{II}=\frac{2\pi M^2g^4}{\mu^4}\left\{\frac{E_1^2-3\mu^2}{2ME_1+\mu^2+M^2}+\frac{M^2(E_1^2-\mu^2)}{(2ME_1+\mu^2+M^2)^2}\right.$$

$$\left.+\frac{8M^2\mu^4}{(4M^3E_1-2\mu^4)(ME_1+\mu^2)-3\mu^4(M^2-\mu^2)}-\frac{\mu^4}{M^2(E_1^2-\mu^2)}\log\right\},$$

$\sigma_{II}(N.R.)$: Same as $PS(PS)$ (units of g^2); $\sigma_{II}(E.R.)=\pi(g/\mu)^4ME_1$, (25)

S(S)

$$\sigma_{II}=\pi\left(\frac{g^2}{\mu}\right)^2\left\{\frac{\mu^2(2ME_1+\mu^2-8M^2)}{4M^2(E_1^2-\mu^2)}\log+\frac{\mu^2(4M^2-\mu^2)^2}{(4M^3E_1-2\mu^4)(ME_1+\mu^2)-3\mu^4(M^2-\mu^2)}\right\}, \tag{26}$$

$\sigma_{II}=(N.R.)=4\pi(g^2/\mu)^2(1+\mu/M)^{-2}$; $\sigma_{II}(E.R.)=\pi g^4(2ME_1)^{-1}\log(2E_1M^{-1})$,

S(V) σ_{II} and σ_I are identical and given by (22).

In the charge exchange case, III, let us for definiteness consider the reaction $\pi^-+P\rightarrow\pi^0+N$. Examination of (7) shows that the total cross section may be written in the form

$$\sigma_{III}=(g_N/g)^2\sigma_I+(g_P/g)^2\sigma_{II}+\text{Interference term} \tag{27}$$

where the "Interference term", proportional to g_Ng_P arises from the product of $\mathfrak{M}_I$ and $\mathfrak{M}_{II}$ in squaring the matrix element. We shall give below the interference term and the limiting forms of the total cross section for each theory.

PS(PS)

$$\text{Int. term}=2\pi g^2g_Ng_P\left\{\frac{1}{2ME_1+\mu^2+M^2}+\frac{\mu^4}{4M^2(2ME_1+\mu^2)(E_1^2-\mu^2)}\log\right\}, \tag{28}$$

$$\sigma_{III}(N.R.)=\pi(g/M)^2(1+\mu/M)^{-2}(1-(\mu/2M)^2)^{-2}$$
$$\{(g_N+g_P)^2+(\mu/2M)^2(g_N-g_P)^2+(\mu/M)(g_P^2-g_N^2)\},$$

$\sigma_{III}(E.R.)=\pi g^2g_P^2(2ME_1)^{-1}\log(2E_1M^{-1})$,

PS(PV)

$$\text{Int. term}=4\pi\frac{M}{\mu^4}g^2g_Ng_P\left\{\frac{E_1(8M^2+\mu^2)+8M^3+4M\mu^2}{2ME_1+\mu^2+M^2}+\frac{2M\mu^2(2M^2-\mu^2)}{(2ME_1+\mu^2)(2ME_1+\mu^2+M^2)}\right.$$

$$-\frac{(2ME_1+\mu^2+8M^2)[ME_1^2+(M^2+\mu^2)E_1+M\mu^2]}{(2ME_1+\mu^2+M^2)^2}$$

$$-\frac{8M^4[ME_1^2+(M^2+\mu^2)E_1+M\mu^2]}{(2ME_1+\mu^2+M^2)^2(2ME_1+\mu^2)} \tag{29}$$

$$\left.-\frac{1}{M(E_1^2-\mu^2)}\left[2M^3E_1+M^2\mu^2-\frac{1}{2}\mu^4-\frac{2M^2\mu^4}{2ME_1+\mu^2}\right]\log\right\},$$

$\sigma_{III}(N.R.)$: same form as $PS(PS)$ in (28),

$\sigma_{III}(E.R.)$: $\pi g^2(g_N-g_P)^2M\mu^{-4}E_1$,

$S(S)$

$$\text{Int. term} = 2\pi g^2 g_N g_P \left\{ \frac{2ME_1+\mu^2+4M^2}{(2ME_1+\mu^2)(2ME_1+\mu^2+M^2)} - \frac{2ME_1-\mu^2+4M^2-(\mu^2/2M)^2}{(E_1^2-\mu^2)(2ME_1+\mu^2)} \log \right\}, \tag{30}$$

$$\sigma_{III}(N.R.) = \frac{4\pi g^2 (g_N-g_P)^2}{\mu^2(1+\mu/M)^2}; \quad \sigma_{III}(E.R.) = \frac{\pi g^2 g_P^2}{2ME_1} \log \frac{2E_1}{M},$$

$S(V)$ Total cross section $=\sigma_{III}=(g_N-g_P)^2 g^{-2}\sigma_I$

where σ_I is given by (22).

(d) *Numerical Results*

A study of the expressions in (b) and (c) for the angular distributions and total scattering cross sections shows some important differences in the predictions made by the various meson theories (2). These differences are most vividly presented by tabulating some of the results for different energies, E_1, of the incident meson. For the scattering processes I and II, results are given in Tables I and II, respectively, for the total cross sections and the ratio of 0° to 90° to 180° scattering in the laboratory system for the meson energies $E_1=1, 3, 10\mu$. Table III refers to the charge exchange scattering, III. We have taken $\mu=286$ electron masses.[10]

Table I. Lowest Order Scattering $\pi^- + P$ or $\pi^+ + N$

	E_1	1μ	3μ	10μ
$PS(PS)$	σ/g^4 $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	$8.9\times10^{-28}\,\text{cm}^2$ 1 : 0.74 : 0.53	5.3×10^{-28} 1 : 0.32 : 0.14	2.1×10^{-28} 1 : 0.059 : 0.014
$S(S)$	σ/g^4 $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1.7×10^{-25} 1 : 0.74 : 0.53	1.8×10^{-26} 1 : 0.45 : 0.24	1.6×10^{-27} 1 : 0.13 : 0.047
$PS(PV)$	σ/g^4 $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	8.9×10^{-28} 1 : 0.74 : 0.53	6.0×10^{-25} 1 : 0.29 : 0.11	3.3×10^{-24} 1 : 0.057 : 0.013
$S(V)$	σ/g^4 $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1.7×10^{-25} 1 : 0.74 : 0.53	8.2×10^{-25} 1 : 0.32 : 0.14	3.4×10^{-24} 1 : 0.059 : 0.014
	Kinematic Factor $J(0^0):J(90^0):J(180^0)$	1 : 0.74 : 0.53	1 : 0.46 : 0.25	1 : 0.15 : 0.055

The important feature to notice about the total cross sections is their decrease with energy E_1 for the $PS(PS)$ and $S(S)$ theories, in contrast to their increase with E_1 for the $PS(PV)$ and $S(V)$ theories. Cross section increasing with the energy are of course characteristic of all meson theories with derivative couplings to the nucleon.

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10) A more recent value for the mass of the π-meson is 276 m_e (F. M. Smith quoted in H. Bradner UCRL 486). This will change the numbers only slightly.

Table II. Lowest Ooder Scattering $\pi^- + N$ or $\pi^+ + P$

	E_1	1μ	3μ	10μ
$PS(PS)$	σ/g^4	$1.2\times10^{-27}\,\mathrm{cm}^2$	9.8×10^{-28}	6.3×10^{-28}
	$\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1 : 0.74 : 0.53	1 : 0.65 : 0.44	1 : 0.38 : 0.22
$S(S)$	σ/g^4	1.7×10^{-25}	1.7×10^{-26}	1.2×10^{-27}
	$\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1 : 0.74 : 0.53	1 : 1.09 : 1.01	0.071 : 0.90 : 1
$PS(PV)$	σ/g^4	1.2×10^{-27}	6.0×10^{-25}	3.3×10^{-24}
	$\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1 : 0.74 : 0.53	1 : 0.29 : 0.12	1 : 0.057 : 0.013
$S(V)$	σ/g^4	1.7×10^{-25}	8.2×10^{-25}	3.4×10^{-24}
	$\sigma(0^0):\sigma(90^0):\sigma(180^0)$	1 : 0.74 : 0.53	1 : 0.32 : 0.14	1 : 0.059 : 0.014
	Kinematic Factor $J(0^0):J(90^0):J(180^0)$	1 : 074 : 0.53	1 : 0.46 : 0.25	1 : 0.15 : 0.055

For process I the decreasing cross section given by the $S(S)$ theory shows a more rapid downward trend than for the $PS(PS)$ theory. This follows from the fact that these theories give cross sections which, in units of g^4, are asymptotically equal at high energies ($\sigma\sim1/E_1$), while they differ markedly at low energies. In the Thomson limit, $E_1=\mu$, the $PS(PS)$ cross section (in units of g^4) is of the order $1/M^2$ whereas the $S(S)$ cross section starts from a much larger value, of the order $1/\mu^2$. The $PS(PV)$ and $S(V)$ cross sections in the Thomson limit are respectively equal to the $PS(PS)$ and $S(S)$ values. At high energies ($E_1>3\mu$) they become asymptotically equal in units of $g^4(\sigma\sim E_1)$. The increase of the $PS(PV)$ cross section with energy near $E_1=\mu$ is therefore very much more rapid than is the case with the $S(V)$ theory. For process II the total cross sections show the same general trends as for I, the main difference being the $(1/E_1)\log E_1$ behavior in the extreme relativistic limit for the $PS(PS)$ and $S(S)$ theories, instead of the $1/E_1$ behavior as in process I.

In the interpretation of the angular distributions shown in Table I and II, it is useful to have the purely kinematic part, common to all theories, separated out. The last line of each table gives this kinematic factor at 0°, 90°, and 180° and represents the angular distribution in the laboratory system that would be produced by an isotropic scattering in the *C.M.* system. In the Thomson limit all theories predict isotropic scattering in the *C.M.* system. For process I the $S(S)$ theory gives a practically isotropic *C.M.* scattering in the energy range from $E_1=\mu$ to $E_1=10\mu$. The square of the $S(S)$ matrix element, $|H|^2_{\text{I}}$, given in eq. (9), is proportional to $A+B+C\approx4+(4/M)E_1+E_1E_2/M^2$, from (15). The dependence on the energy of the scattered meson, E_2, gives a preferential forward scattering in the *C.M.* systems but this does not cause appreciable departure from isotropy until E_1 becomes comparable with M. The deviations from isotropy begin to be noticeable at $E_1=10\mu$ as we see from Table I. For process II the

$S(S)$ theory gives an increasingly strong backward scattering in the *C.M.* system; at $E_1=10\mu$ the effect is large enough to cause a strong backward scattering even in the laboratory system. The reason for this behavior is seen (cf. (10) and (15)) to come from the factor $(2ME_2-\mu^2)^{-2}$, which is most effective in favoring backward scattering, and also from the factor $A+B-C\approx 4-(4/M)E_2+E_1E_2/M^2$, which favors backward scattering until E_1 is larger than $4M$.

In contrast to the $S(S)$ theory, we find a strong forward scattering in the *C.M.* system at high energies predicted for process I by the $PS(PS)$ theory. For process II the $PS(PS)$ theory gives backward scattering in the *C.M.* system but the effect is not as pronounced as for the $S(S)$ theory due to a competition between the factors A and $(2ME_2-\mu^2)^{-2}$ in eq. (10). The $S(V)$ theory gives an angular distribution which is the same for processes I and II and is exactly identical with that of the $PS(PS)$ theory for process I. The $PS(PV)$ theory is indistinguishable from $S(V)$ on the basis of angular distribution alone.

To summarize, for scattering of type I three of the theories (2) give essentially identical results in predicting marked forward scattering in the *C.M.* system, while the remaining theory, $S(S)$, predicts an isotropic *C.M.* scattering in the energy range $\mu\leqq E_1\leqq 10\mu$. More diversity is found for the scattering of type II with two of the theories, $PS(PV)$ and $S(V)$ predicting marked forward scattering and the remaining ones predicting backward *C.M.* scattering, more pronounced for the $S(S)$ than for $PS(PS)$.

Finally we consider the charge exchange scattering, process III. Without knowing the relative magnitudes and signs of the mesic charges g_N and g_P of neutron and proton, respectively, for interaction with neutral mesons, it is not possible to state the results of the different theories in terms of a single coupling parameter. Instead we give the total cross sections for $E_1=1, 3, 10\mu$ in terms of g, g_N, and g_P, fixing our attention on the reaction $\pi^-+P\rightarrow\pi^0+N$.

$PS(PS)$

$$\begin{aligned}
E_1&=\mu & \sigma_{III}&=g^2[8.93g_N^2+12.22g_P^2+20.90g_Ng_P]\times 10^{-28}\,\mathrm{cm}^2,\\
E_1&=3\mu & \sigma_{III}&=g^2[5.26g_N^2+\ \ 9.80g_P^2+14.17g_Ng_P]\times 10^{-28},\\
E_1&=10\mu & \sigma_{III}&=g^2[2.06g_N^2+\ \ 6.30g_P^2+\ \ 6.69g_Ng_P]\times 10^{-28},
\end{aligned}\tag{32}$$

$S(S)$

$$\begin{aligned}
E_1&=\mu & \sigma_{III}&=g^2(g_N-g_P)^2 1.70\times 10^{-25}\,\mathrm{cm}^2,\\
E_1&=3\mu & \sigma_{III}&=g^2[1.85g_N^2+1.66g_P^2-3.43g_Ng_P]\times 10^{-26},\\
E_1&=10\mu & \sigma_{III}&=g^2[1.59g_N^2+1.24g_P^2-2.09g_Ng_P]\times 10^{-27},
\end{aligned}\tag{33}$$

$PS(PV)$

$$\begin{aligned}
E_1&=\mu & \sigma_{III}&:\ \text{Same as } PS(PS) \text{ in units of } g^2,\ g_N^2,\ g_P^2,\\
E_1&=3\mu & \sigma_{III}&=g^2[6.02g_N^2+6.00g_P^2+2.54g_Ng_P]\times 10^{-25}\,\mathrm{cm}^2,\\
E_1&=10\mu & \sigma_{III}&=g^2[3.31g_N^2+3.31g_P^2-0.62g_Ng_P]\times 10^{-24},
\end{aligned}\tag{34}$$

$S(V)$ $\qquad \sigma_{III}=(g_N-g_P)^2g^{-2}\sigma_I$; σ_I given by (22). (35)

The dependence on $(g_N-g_P)^2$ shown by the $S(V)$ theory in (35) is approximately valid for the $S(S)$ theory as well, in the energy range considered. The $PS(PS)$ theory shows roughly a dependence upon $(g_N+g_P)^2$ in the region of small kinetic energy for the meson, $\mu \leq E_1 \leq 2\mu$, but a more complicated behavior for larger energies. The $PS(PV)$ theory starts as $(g_N+g_P)^2$ at very small meson kinetic energy and makes a transition to a $(g_N-g_P)^2$ dependence at very high energies.

To give an idea of how radically the angular distribution and total cross section of the charge exchange scattering depends upon the choice of g_N and g_P, we have considered the two cases $g_N=g_P$ and $-g_N=g_P$. The results are given

Table III. Charge Exchange Scattering: $\pi^- + P \rightarrow \pi^0 + N$
or
(Lowest Order) $\pi^+ + N \rightarrow \pi^0 + P$

$\boxed{g_N=g_P}$

	E_1	1μ	3μ	10μ
$PS(PS)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	4.2×10^{-27} cm^2 1 : 0.74 : 0.53	2.9×10^{-27} 1 : 0.48 : 0.27	1.5×10^{-27} 1 : 0.19 : 0.088
$S(S)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	0 ——	8.3×10^{-28} 1 : 0.016 : 0.22	7.5×10^{-28} 1 : 0.036 : 0.082
$PS(PV)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	4.2×10^{-27} 1 : 0.74 : 0.53	1.5×10^{-24} $10^{-5}:1:10^{-5}$	6.0×10^{-24} $10^{-7}:1:10^{-7}$
$S(V)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	0 ——	0 ——	0 ——
	Kinematic Factor $J(0^0):J(90^0):J)180^0)$	1 : 0.74 : 0.53	1 : 0.46 : 0.25	1 : 0.15 : 0.055

$\boxed{-g_N=g_P}$

	E_1	1μ	3μ	10μ
$PS(PS)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	2.6×10^{-29} 1 : 0.74 : 0.53	8.9×10^{-29} 0.027 : 0.73 : 1	1.7×10^{-28} 0.002 : 1.14 : 1
$S(S)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	6.8×10^{-25} 1 : 0.74 0.53	6.9×10^{-26} 1 : 0.67 : 0.48	4.9×10^{-27} 1 : 0.50 : 0.37
$PS(PV)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	2.6×10^{-29} 1 : 0.74 : 0.53	9.5×10^{-25} $1:10^{-4}:0.11$	7.2×10^{-24} $1:10^{-5}:0.013$
$S(V)$	$\sigma/g^2g_0^2$ $\sigma(0^0):\sigma(90^0):\sigma(180^0)$	5.8×10^{-25} 1 : 0.74 : 0.53	3.3×10^{-24} 1 : 0.32 : 0.14	1.4×10^{-23} 1 : 0.059 : 0.014
	Kinematic Factor $J(0^0):J(90^0):J(180^0)$	1 : 0.74 : 0.53	1 : 0.46 : 0.25	1 : 0.15 : 0.055

in Table III. It should be emphasized that the neglect of the mass difference between neutron and proton and between neutral and charged mesons is partially responsible for the very extreme angular variations shown, for example, by the $PS(PV)$ theory when we set g_N and g_P equal in absolute value. A more exact calculation should still yield trends similar to those shown in Table III. In particular the dependence of the total cross section in the $S(V)$ theory on $(g_N-g_P)^2$ is preserved even if the mass differences are accounted for.

§ 3. Higher Order Corrections to the Scattering

In this section we shall consider corrections to the lowest order scattering matrix element which are of fourth order in the meson-nucleon coupling constant g. These corrections produce additional terms in the cross section of order g^6, or of order g^2 relative to the lowest order cross section. The divergences inherent in such corrections are removable by the technique of mass and charge renormalization only in the $PS(PS)$ and $S(S)$ theories from among the four theories considered previously in (2). In the $PS(PV)$ and $S(V)$ theories it is not possible to generalize the renormalization procedures of electrodynamics because of the presence of derivatives of the meson wave function and hence extra powers of the meson momentum in integrations over the momentum space. We therefore restrict the discussion to the $PS(PS)$ and $S(S)$ theories. Because of analytical complications the corrections are calculated only in the Thomson limit. In the following we shall first list the different types of Feynman diagrams appropriate to the higher order scattering, follow this by a separate discussion of the self-energy of a free nucleon and of the renormalization of the meson-nucleon coupling constant, and finally shall apply these results to obtain numerical values for the g^4 correction terms in the matrix element for the scattering processes I, II and III.

To obtain the higher order scattering diagrams we may start with the lowest

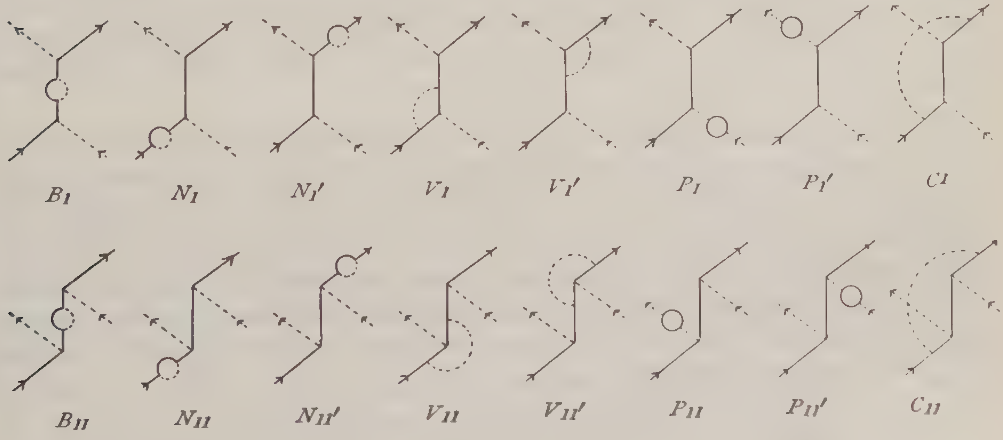

Fig. 2. Feynman diagrams for the fourth order meson-nucleon scattering.

order diagrams of Fig. 1 as a framework and insert virtual meson or nucleon lines wherever this is consistent with charge conservation. Fig. 2 contains the complete list of diagrams which contribute g^4 terms in the matrix element. They are divided into categories I and II according to the diagram of Fig. 1 from which they are obtained, and as an aid to the memory, are labeled with letters like B for "bubble" term, V for vertex term, P for vacuum polarization term (analogous to the electromagnetic case), N for normalization term, an C for convergent term. Only the matrix elements corresponding to the C diagrams give convergent integrals over the momentum space. All the others contain infinite parts which we shall show can be interpreted in terms of corrections to the nucleon mass and mesic charge, g.

(a) *The Nucleon Self Energy*

A free nucleon of momentum $\boldsymbol{p}$ (with spinor u, satisfying $\boldsymbol{p}u=Mu$; $\bar{u}u=1$) may virtually emit a meson of momentum $\boldsymbol{k}$, propagate in a virtual state of momentum $\boldsymbol{p}-\boldsymbol{k}$, and subsequently reabsorb the same meson restoring itself to its original momentum state. The resulting change in the amplitude for a free nucleon, in momentum state $\boldsymbol{p}$, to arrive at a later time in the same momentum state, is expressible in terms of a correction ΔM to the mass of the free nucleon. The analysis of Feynman shows that for the $PS(PS)$ theory ΔM is the diagonal element between $\bar{u}$ and u of the "self-energy matrix"

$$i(g^2/\pi)\int\gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5 d^4k(\boldsymbol{k}^2-\mu^2)^{-1}$$

where d^4k means $(2\pi)^{-2}dk_1dk_2dk_3dk_4$. This differs in sign from the electromagnetic self energy matrix after replacing the γ_μ in the electromagnetic interaction by γ_5. Such a change of sign is characteristic of the pseudoscalar or scalar meson-nucleon interactions and is readily obtained from Feynman's Lagrangian formulation of quantum mechanics applied to the nucleon interacting with the infinitely many meson field oscillators.[11] In general there is a factor ig^2/π for each virtual meson in any integration over the momentum of such a meson, provided the above convention for d^4k is followed.

To avoid ambiguities in handling formal expressions which are infinite in value, as is the case with the self energy, we shall follow Feynman in the introduction of a convergence factor $C(k^2-\mu^2)=-(\lambda^2-\mu^2)(k^2-\lambda^2)^{-1}$ with $\lambda \gg M$ multiplying the $d^4k(k^2-\mu^2)^{-1}$ in the integration over a virtual meson. This makes finite the self-energy and all quantities involving integration over virtual mesons. Identification of the self-energy contribution to more complicated matrix elements like the bubble terms of Fig. 2, is then a simple matter, free of misinterpretations. The self-energy matrix now has the form

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[11] R. P. Feynman, Rev. Mod. Phys. **20** (1948), 367.

$$i(g^2/\pi)\int\gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5(k^2-\mu^2)^{-1}d^4k\,C(k^2-\mu^2). \tag{36}$$

Evaluation of this integral in the Appendix (a), and application to the state of a free nucleon of momentum $\boldsymbol{p}$ gives for the g^2 correction to the mass

$PS(PS)$

$$\Delta M=(g^2/8\pi)M\{2\log(\lambda/M)-\frac{1}{2}+\omega+\omega(1-\frac{\omega}{2})\log\omega-\omega\sqrt{\tau}\,T(\omega)\} \tag{37}$$

where

$$\omega=(\mu/M)^2: \qquad \tau=4\omega-\omega^2,$$

$$T(\omega)=\tan^{-1}\frac{\omega}{\sqrt{\tau}}+\tan^{-1}\frac{2-\omega}{\sqrt{\tau}}. \tag{37$'$}$$

The nucleon self-energy matrix for the $S(S)$ theory is obtained from (36) by replacing γ_5 with unity. We find

$S(S)$

$$\Delta M=-(g^2/8\pi)M\{6\log(\lambda/M)+\frac{9}{2}-\omega-\omega(3-\frac{1}{2}\omega)\log\omega-(4-\omega)\sqrt{\tau}\,T(\omega)\}. \tag{38}$$

(b) *Renormalization of the Mesic Charge of the Nucleon*

A measure of the strength of the meson-nucleon coupling is most directly obtained by considering a process in which the nucleon of interest interacts only once with the meson field, yielding a matrix element proportional to the first power of the coupling constant g. Renormalization of g comes from a consideration of the higher order corrections to such a first order process.

We therefore examine the scattering of a free nucleon by a meson potential; this corresponds to the diagram (i) of Fig. 3, in which a nucleon of initial momentum $\boldsymbol{p}$ is scattered to a final state of momentum $\boldsymbol{p}+\boldsymbol{q}$ as a result of interaction with a meson potential varying in space and time as $\phi\exp(-iq\cdot x)$. The lowest order matrix element is

$$g\mathfrak{M}_0=g\phi(\bar{u}_2\gamma_5u_1) \tag{39}$$

where we have written the result for the $PS(PS)$ theory and denoted by u_1 and u_2 the spinors for the initial and final states of the nucleon, respectively.

Corrections to the lowest order result are provided by the remaining diagrams (ii), (iii), (iv) and (v) of Fig. 3. Because of the necessity to distinguish between neutron and proton in their interactions with neutral mesons, let us for definiteness suppose that ϕ represents a charged meson field. The diagrams (ii) and (iii) correspond to what Feynman calls "renormalization terms" and if we assume that we are dealing with virtual charged mesons in the two nucleon arms of (ii) and (iii), these diagrams together yield a matrix element simply proportional to the lowest order element:

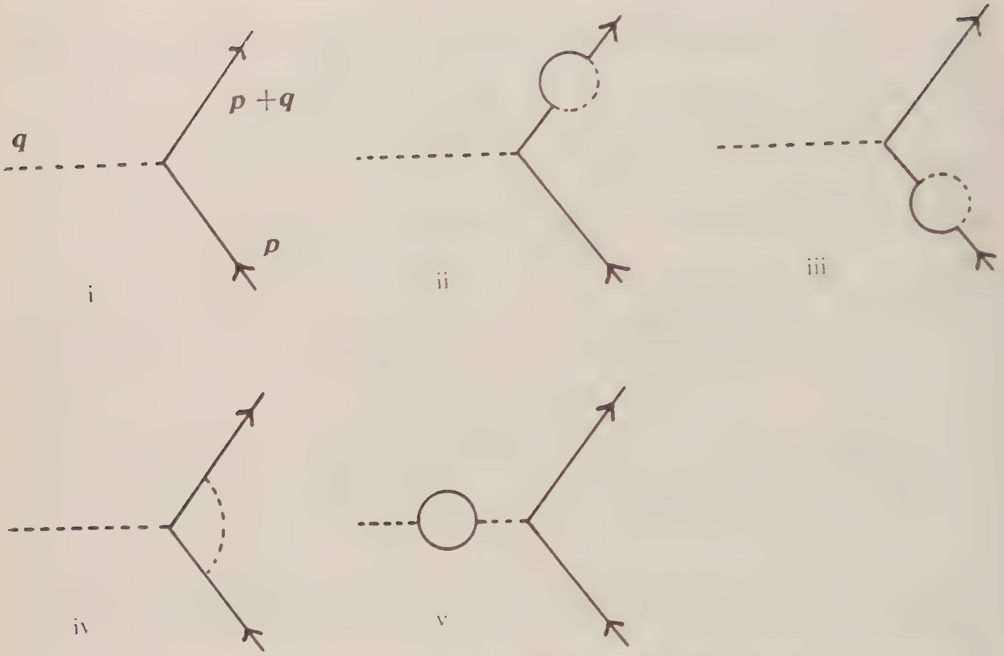

Fig. 3. Feynman diagrams for the scattering of a nucleon by a meson potential. The lowest order diagram is shown in (i). Corrections due to emission and reabsorption of a virtual meson are shown in (ii), (iii), and (iv). Creation of a virtual nucleon-antinucleon pair by the meson potential produces the effect shown in (v).

$$\mathfrak{M}_r = -rg^2 \cdot g\mathfrak{M}_0 \tag{40}$$

where the factor r is defined by[12]

$$r = (i/\pi)\int \gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\boldsymbol{p}M^{-1}(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5(\boldsymbol{k}^2-\mu^2)^{-1}d^4k\,C(\boldsymbol{k}^2-\mu^2). \tag{40$'$}$$

After the integration is performed, $\boldsymbol{p}$ is to be replaced everywhere by M. For the $PS(PS)$ theory we obtain, according to Appendix (b),

$$r_{PS} = (1/8\pi)\{2\log(\lambda/M) + \frac{1}{2} + 3\omega + \omega(2-\frac{3}{2}\omega)\log\omega - (\omega^2/\sqrt{\tau})(10-3\omega)\,T(\omega)\}. \tag{40$''$}$$

Since r is completely independent of the scattering process, it is clear that the renormalization terms may be considered to provide a correction to the coupling constant of amount $-rg^2 \cdot g$. If we allow virtual neutral mesons in the two nucleon arms of (ii) and (iii), the correction to g becomes $-(1/2)r(g_N^2+g_c^2)\cdot g$. The $S(S)$ theory gives the result

$$r_S = r_{PS} - (1/8\pi)\{8+4(1-\omega)\log\omega - 8(\omega/\sqrt{\tau})(3-\omega)\,T(\omega)\}. \tag{41}$$

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[12] Reference 4, page 778.

Diagram (iv) of Fig. 3, the "vertex term" in the scattering, contributes only if the virtual meson cutting across the vertex is neutral. (ϕ is still chosen to represent a charged meson field). The vertex contribution to the matrix for the scattering is then, in the $PS(PS)$ theory,

$$g\gamma_5(ig_N g_P/\pi)\int(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}-M)^{-1}\gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5(\boldsymbol{k}^2-\mu^2)^{-1}d^4kC(\boldsymbol{k}^2-\mu^2). \quad (42)$$

From (9b) of the appendix, where the required integration is performed, it is found that the vertex term contributes the matrix element

$$\mathfrak{M}_V = g_N g_P v(\boldsymbol{q}^2) g\mathfrak{M}_0 \quad (43)$$

where for our present purposes it is important to note that the multiplying factor $v(\boldsymbol{q}^2)$ depends upon $\boldsymbol{q}^2$ alone and is sensitive to the cut-off parameter λ; $v(\boldsymbol{q}^2)\rightarrow\infty$ for $\lambda\rightarrow\infty$. The part of $v(\boldsymbol{q}^2)$ which is sensitive to λ may be absorbed as a correction to g in the following manner.* Let $v(\boldsymbol{q}^2)$ be developed in a power series about the value $\boldsymbol{q}^2=\mu^2$:

$$v(\boldsymbol{q}^2)=v(\mu^2)+c(\boldsymbol{q}^2-\mu^2)+d(\boldsymbol{q}^2-\mu^2)^2+\cdots\cdots$$

where the coefficients c, d,$\cdots\cdots$are easily seen to be independent of the cut-off. Now the meson potential ϕ in (39) can be written in the form $\phi=(\boldsymbol{q}^2-\mu^2)^{-1}\rho$ where $(\boldsymbol{q}^2-\mu^2)^{-1}$ is the propagation factor for the virtual meson $\boldsymbol{q}$ in the Fourier decomposition of the meson potential. Alternatively, the relation $(\boldsymbol{q}^2-\mu^2)\phi=\rho$ may be regarded as the expression in momentum space of the meson field equation connecting the potential with its sources, ρ. As a result the vertex contribution is proportional to

$$v(\boldsymbol{q}^2)\phi=v(\mu^2)\phi+c\rho+d(\boldsymbol{q}^2-\mu^2)\rho+\cdots\cdots$$

The first term in this series, and the only one sensitive to λ, is a constant times the potential ϕ; the remaining finite terms are independent of ϕ and are properly to be interpreted as higher order corrections to the scattering. From the term $v(\mu^2)\phi$ we see that the vertex diagram contributes the correction $g_N g_P v(\mu^2)\cdot g$ to the coupling constant g. After performing the integrations in (10, 11b) of the appendix we find

$$v_{PS}(\mu^2)=(8\pi)^{-1}\{-4\log(\lambda/M)+2.09\}, \quad (44)$$

$$v_S(\mu^2)=(8\pi)^{-1}\{-4\log(\lambda/M)+11.39\}. \quad (45)$$

Diagram (v) of Fig. 3 corresponds to the creation of a virtual nucleon-antinucleon pair by the meson potential and is analogous to the electromagnetic vacuum polarization effect on the scattering of an electron by a potential. The matrix element which it contributes is of the form $2g^2(\boldsymbol{q}^2-\mu^2)^{-1}J(\boldsymbol{q}^2)g\mathfrak{M}_0$ where $J(\boldsymbol{q}^2)$ is a divergent integral over the momentum of the virtual nucleon and may be cut-off by the same method as for the electromagnetic case. The factor 2 corresponds to the two possibilities, neutron and proton, for the intermediate nucleon. Expanding $J(\boldsymbol{q}^2)$ about $\boldsymbol{q}^2=\mu^2$, the matrix element is proportional to

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* We wish to thank Dr. K. M. Case for a discussion of this point.

$$(\boldsymbol{q}^2-\mu^2)^{-1}J(\boldsymbol{q}^2)\phi=(\boldsymbol{q}^2-\mu^2)^{-2}J(\mu^2)\rho+J'(\mu^2)\phi+\frac{1}{2}J''(\mu^2)\rho+\cdots\cdots$$

Both $J(\mu^2)$ and its first derivative $J'(\mu^2)$ are divergent (sensitive to the cut-off) but $J''(\mu^2)$ is well defined. The first term in this series can be regarded as correcting the lowest order element $(\boldsymbol{q}^2-\mu^2)^{-1}\rho$ for the fact that the meson mass has an addition $\Delta\mu$ proportional to $J(\mu^2)$, as is readily verified.[13] The second term provides a correction to g^2 of amount $\Delta(g^2)/g^2=2g^2J'(\mu^2)$. (One obtains a correction to g^2 since there is a factor g in the source ρ and the nucleon closed loop is symmetrically placed between source and point of scattering). Therefore $\Delta g=g^2J'(\mu^2)\cdot g$. In the application to the problem in which we are interested, the scattering of free mesons by nucleons, the $J'(\mu^2)$ correction to g will automatically cancel out and it is therefore unnecessary to obtain its explicit expression.

Considering all the diagrams of Fig. 3 together, and allowing both charged and neutral intermediate mesons whenever this is consistent with charge conservation, we see that the part of the total matrix element proportional to the meson potential ϕ is

$$g[1-(g^2+\frac{1}{2}g_N^2+\frac{1}{2}g_P^2)r+g_Ng_Pv(\mu^2)+g^2J'(\mu^2)]\phi(\bar{u}_2\gamma_5u_1).$$

The scattering experiment considered here may thus be used to define an "experimental" coupling constant, g^{exp}, given by

$$g^{\text{exp}}=g[1-(g^2+\frac{1}{2}g_N^2+\frac{1}{2}g_P^2)r+g_Ng_Pv(\mu^2)+g^2J'(\mu^2)]. \tag{46}$$

By taking ϕ to represent a neutral meson field, and allowing the incident nucleon to be alternately a neutron and a proton, one finds corrections to g_N and g_P.

$$g_N^{\text{exp}}=g_N[1-(g^2+g_N^2)r+(g^2+g_N^2)v(\mu^2)+\frac{1}{2}(g_N^2+g_P^2)J'(\mu^2)], \tag{47}$$

$$g_P^{\text{exp}}=g_P[1-(g^2+g_P^2)r+(g^2+g_P^2)v(\mu^2)+\frac{1}{2}(g_N^2+g_P^2)J'(\mu^2)]. \tag{48}$$

(c) *Bubble Terms*

Let us turn next to the higher order correctoins to the meson-nucleon scattering and begin by considering a typical "bubble term" B_I of Fig. 2. Using the same notation for the nucleon and meson momenta as in Fig. 1, the associated matrix element can be written ($PS(PS)$ theory)

$$g^4B_I=-ig^2\bar{u}_2\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}\gamma_5g^2O_I\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}\gamma_5u_1, \tag{49}$$

$$O_I\equiv(i/\pi)\int(\boldsymbol{p}_1+\boldsymbol{k}_1-\boldsymbol{k}-M)^{-1}(\boldsymbol{k}^2-\mu^2)^{-1}d^4kC(\boldsymbol{k}^2-\mu^2). \tag{49$'$}$$

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[13] K. M. Watson and J. V. Lepore, Phys. Rev. **76** (1949), 1157.

The operator $g^2\gamma_5 0_I \gamma_5$ is formally the same as the self energy matrix (36) in the $PS(PS)$ theory but differs in that $(\boldsymbol{p}_1+\boldsymbol{k}_1)^2 \neq M^2$. According to Appendix (c) the operator 0_I has the form

$$0_I = -(8\pi)^{-1}\{(\boldsymbol{p}_1+\boldsymbol{k}_1+M)[4\log(\lambda/M)+U]-(\boldsymbol{p}_1+\boldsymbol{k}_1)[2\log(\lambda/M)+X]\} \tag{50}$$

where U and X are finite terms independent of the cut-off and given explicitly in the Appendix (c). Note that

$$\gamma_5 0_I \gamma_5 = -(8\pi)^{-1}\{(\boldsymbol{p}_1+\boldsymbol{k}_1-M)[2\log(\lambda/M)+U-X]-M[2\log(\lambda/M)+X]\}$$

and that the $(\boldsymbol{p}_1+\boldsymbol{k}_1-M)$ factor in this expression may be regarded as canceling one of the $(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}$ of (49). We may therefore put B_I in the form

$$g^4 B_I = g^4(G\mathfrak{M}_I + F), \tag{51}$$

$$g^4 G \equiv -g^2(g^2/8\pi)[2\log(\lambda/M)+U-X], \tag{51$'$}$$

$$g^4 F \equiv -ig^2\bar{u}_2\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}(g^2M/8\pi)[2\log(\lambda/M)+X](\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}\gamma_5 u_1 \tag{51$''$}$$

where $g^2\mathfrak{M}_I$ is the lowest order matrix element (3). Formally, G and F both become infinite in the limit $\lambda \to \infty$.

The fact that $g^4 G\mathfrak{M}_I$ is proportional to the lowest order matrix element, $g^2\mathfrak{M}_I$, suggests the possible removal of its λ-dependent part through the renormalization of g discussed in the previous section. Let us consider the following matrix elements for process I which involve only charged mesons in the intermediate states: the lowest order element itself, $g^2\mathfrak{M}_I$, the part $g^4 G\mathfrak{M}_I$ of $g^4 B_I$, and the contribution $-rg^4\mathfrak{M}_I$ from the renormalization diagrams N_I and N_I'. Now, according to the renormalization of g obtained in (46), we should express g everywhere in terms of the "experimental" coupling constant, g^{exp}. To the fourth order in g, this makes a difference only in the contribution of the lowest order diagram, $g^2\mathfrak{M}_I$. This is to be written

$$\begin{aligned} g^2\mathfrak{M}_I = {} & (g^{\text{exp}})^2\mathfrak{M}_I + \\ & + (g^{\text{exp}})^2\mathfrak{M}_I\{(2g^2+g_N^2+g_P^2)r - 2g_N g_P v(\mu^2) - 2g^2 J'(\mu^2)\}. \end{aligned} \tag{52}$$

The second line of (52) represents a fourth order correction which must be added to all the other corrections obtained; to this order of accuracy it is not essential to distinguish between $g^2(g^{\text{exp}})^2$ and g^4. These corrections in (52) will subsequently be used, one by one, to eliminate the divergences inherent in the contributions from the diagrams of Fig. 2. Thus by adding the correction $2rg^4\mathfrak{M}_I$ of (52) to the part $g^4G\mathfrak{M}_I$ of g^4B_I and $-rg^4\mathfrak{M}_I$ from diagrams N_I and N_I', we obtain the net contribution

$$g^4(G+r)\mathfrak{M}_I. \tag{53}$$

Comparison of G in (51) with r in (40)$''$ shows that the contribution (53) is finite, independent of the cut-off. For a neutral virtual meson in the diagram B_I, it is necessary to add $-rg_P^2g^2\mathfrak{M}_I$ for N_I and N_I', together with the correction $r(g_N^2+g_P^2)g^2\mathfrak{M}_I$ from (52). The net contribution is $g_N^2(G+r)g^2\mathfrak{M}_I$. There is no difficulty in extending these considerations to the charge exchange reactions III. It is only necessary to use the expression (47) and (48) for the renormalized coupling constants g_N and g_P.

The tèrm g^4F in (51) leads to a finite correction after taking into account the charge, ΔM, in the nucleon mass discussed in (a). The "experimental" mass of the nucleon is $M^{\text{exp}}=M+\Delta M$, and we should express M everywhere as $M^{\text{exp}}-\Delta M$. To the fourth order in g, this makes a difference only in the lowest order matrix element, $(g^{\text{exp}})^2\mathfrak{M}_I$ of (52). Substitution of $M^{\text{exp}}-\Delta M$ for M in the propagation factor $(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}$ and treatment of ΔM as a small perturbation, gives

$$(g^{\text{exp}})^2\mathfrak{M}_I=-i(g^{\text{exp}})^2\bar{u}_2\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M^{\text{exp}})^{-1}\gamma_5u_1+$$
$$+i(g^{\text{exp}})^2\bar{u}_2\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M^{\text{exp}})^{-1}\Delta M(\boldsymbol{p}_1+\boldsymbol{k}_1-M^{\text{exp}})^{-1}\gamma_5u_1. \qquad (54)$$

In the fourth order correction represented by the second line of (54), the M^{exp} may be replaced by M and the g^{exp} by g. When this correction is added to the g^4F term of (51)″, a result is obtained proportional to

$$(g^2M/8\pi)(2\log(\lambda/M)+X)-\Delta M. \qquad (55)$$

The expression (37) for ΔM shows that (55) is independent of the cut-off.

To summarize, we obtain a net finite contribution from the bubble diagram B_I and the renormalization diagrams N_I and N_I' by taking account of the nucleon mass and charge renormalization. This finite bubble contribution is characterized by a single number, which we shall henceforth call simply B_I^0. Appropriate powers of g, g_N, and g_P multiplying B_I^0 will give the bubble effects for the different reactions $\pi^-+P\to\pi^-+P$ or $\pi^-+P\to\pi^0+N$, etc.

Entirely similar considerations apply to the diagrams B_{II}, N_{II}, and N_{II}' of Fig. 2, yielding a characteristic number B_{II}^0. These number B_I^0 and B_{II}^0, for the Thomson limit, are listed in Table IV, for the two theories $PS(PS)$ and $S(S)$. Details are to be found in Appendix (c).

(d) *Vertex Terms*

For the scattering process $\pi^-+P\to\pi^-+P$, the diagram V_I of Fig. 2 provides a matrix element ($PS(PS)$ theory)

$$-ig^2\bar{u}_2\gamma_5(\boldsymbol{p}_1+\boldsymbol{k}_1-M)^{-1}\gamma_5g_Ng_P\Lambda_Iu_1 \qquad (56)$$

with

$$\Lambda_I=(i/\pi)\int(\boldsymbol{p}_1+\boldsymbol{k}_1-\boldsymbol{k}-M)^{-1}\gamma_5(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}\gamma_5(\boldsymbol{k}^2-\mu^2)^{-1}d^4kC(\boldsymbol{k}^2-\mu^2). \quad (56)'$$

Evaluation of this integral (Appendix (d)) gives

$$\Lambda_I = -(2\pi)^{-1}\log(\lambda/M) + \text{finite terms.} \tag{57}$$

Addition to (56) of the correction $-g^2 g_N g_P v(\mu^2)\mathfrak{M}_I$ from the charge renormalization terms in (52), and comparison of (44) with (57) shows thar the net vertex contribution from V_I is finite and may be written $g^2 g_N g_P V_I^0$ where V_I^0 is a number independent of the coupling constants. A similar result holds for diagrams V_I', V_{II}, and V_{II}' of Fig. 2. In the Thomson limit $V_I^0 = V_I'^0$ and $V_{II}^0 = V_{II}'^0$. Numerical results are given in Table IV.

(e) *Convergent Terms*

It is clear that all effects due to the "polarization" diagrams P_I, P_I', P_{II}, and P_{II}' are exactly cancelled out by the polarization terms in (52). We are therefore left with the "convergent" diagrams C_I and C_{II}. For the pseudoscalar case, the diagram C_I is represented by the convergent integral

$$C_I = \bar{u}_2(1/\pi)\int \gamma_5(\boldsymbol{p}_2 - \boldsymbol{k} - M)^{-1}\gamma_5(\boldsymbol{q} - \boldsymbol{k} - M)^{-1}\gamma_5(\boldsymbol{p}_1 - \boldsymbol{k} - M)^{-1}\gamma_5(\boldsymbol{k}^2 - \mu^2)^{-1} d^4k \cdot u_1 \tag{58}$$

with the abbreviation $\boldsymbol{q} = \boldsymbol{p}_1 + \boldsymbol{k}_1$. The convergence factor $C(\boldsymbol{k}^2 - \mu^2)$ which has been used previously to cut off divergent integrals is not necessary. After rationalizing the propagation factors and eliminating the γ_5's, C_I takes the form

$$C_I = \bar{u}_2(1/\pi)\int \frac{\boldsymbol{k}(\boldsymbol{q} - \boldsymbol{k} + M)\boldsymbol{k} d^4k}{[\boldsymbol{k}^2 - 2p_2 \cdot k][(\boldsymbol{q} - \boldsymbol{k}^2 - M^2][\boldsymbol{k}^2 - 2p_1 \cdot k][\boldsymbol{k}^2 - \mu^2]} u_1. \tag{59}$$

Replacement of $\boldsymbol{q}$ by $\boldsymbol{p}_1 - \boldsymbol{k}_2$ in (59) gives the integral C_{II}. Evaluation of these integrals in the general case of arbitrary meson energies has proved too difficult. In the Thomson limit the integration is carried out by the methods outlined in Appendix (e) and the results given in Table IV.

(f) *Compilation of Results*

All the results for the higher order matrix elements in the Thomson limit are assembled in Table IV: net contribution of bubble term B^0 (types I and II), net contribution of vertex term V^0, convergent contribution C. For comparison the matrix elements in lowest order are also listed.

$$PS(PS) \qquad \mathfrak{M}_I = -i(2M+\mu)^{-1}; \qquad \mathfrak{M}_{II} = -i(2M-\mu)^{-1}, \tag{60}$$

$$S(S) \qquad \mathfrak{M}_I = -i/\mu; \qquad \mathfrak{M}_{II} = i/\mu. \tag{61}$$

(It is evidently convenient to multiply all $PS(PS)$ matrix by $2Mi$ and all $S(S)$ matrix elements by μi).

To obtain the higher order matrix elements for the various scattering processes, it is necessary to multiply the numbers in Table IV by appropriate coefficients containing g, g_N, g_P and sum. These coefficients are listed in Table V.

From the numbers in Table IV it is clear that the additional terms in the scattering cross section will be of order unity times g^2 relative to the lowest order σ_0. As an example, if we assume $-g_N=g_P=g_0$, we obtain;

$$PS(PS)\qquad \begin{aligned} \sigma_I/\sigma_0&=1-0.43\,g^2-0.03\,g_0^2,\\ \sigma_{II}/\sigma_0&=1-0.64\,g^2+0.16\,g_0^2,\ ^{14)}\\ \sigma_{III}/\sigma_0&=1-0.30\,g^2-1.88\,g_0^2,\end{aligned} \tag{62}$$

$$S(S)\qquad \begin{aligned} \sigma_I/\sigma_0&=1-1.27\,g^2-4.30\,g_0^2,\\ \sigma_{II}/\sigma_0&=1+1.12\,g^2+0.67\,g_0^2,\\ \sigma_{III}/\sigma_0&=1-0.94\,g^2-0.07\,g_0^2\end{aligned} \tag{63}$$

Table VI.
Comparison of Lowest Order and Higher Order Matrix Elements.
Thomson Limit.

PS(PS)

Type of Diagram	Lowest Order	Higher Order Matrix Elements		
	$(2Mi)\mathfrak{M}$	$(2Mi)B^0$	$(2Mi)V^0$	$(2Mi)C$
I	0.928	−0.110	−0.203	−0.309
II	1.085	−0.039	−0.108	−0.091

S(S)

Type of Diagram	Lowest Order	Higher Order Matrix Elements		
	$(i\mu)\mathfrak{M}$	$(i\mu)B^0$	$(i\mu)V^0$	$(i\mu)C$
I	1	−0.575	0.557	−0.461
II	−1	−0.099	0.089	−0.059

Table V.
Coefficients for Correction Terms

Process	B_I	B_{II}	V_I	V_{II}	C_I	C_{II}
$\pi^-+P\to\pi^-+P$	$g^2(g^2+g_N^2)$	0	$2g^2g_Ng_P$	0	$g^2g_P^2$	g^4
$\pi^++N\to\pi^++N$	$g^2(g^2+g_P^2)$	0	$2g^2g_Ng_P$	0	$g^2g_N^2$	g^4
$\pi^-+N\to\pi^-+N$	0	$g^2(g^2+g_P^2)$	0	$2g^2g_Ng_P$	g^4	$g^2g_N^2$
$\pi^++P\to\pi^++P$	0	$g^2(g^2+g_N^2)$	0	$2g^2g_Ng_P$	g^4	$g^2g_P^2$
$\pi^-+P\to\pi^0+N$	$gg_N(g^2+g_N^2)$	$gg_P(g^2+g_P^2)$	$gg_N^2(g_N+g_P)$ $+$ g^3g_P	$gg_P^2(g_N+g_P)$ $+$ g^3g_N	$gg_Pg_N^2$	$gg_Ng_P^2$
$\pi^++N\to\pi^0+P$	$gg_P(g^2+g_P^2)$	$gg_N(g^2+g_N^2)$	$gg_P^2(g_N+g_P)$ $+$ g^3g_N	$gg_N^2(g_N+g_P)$ $+$ g^3g_P	$gg_Ng_P^2$	$gg_Pg_N^2$

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14) For process II, *PS(PS)*, and $g_0=0$, Corinaldesi and Field (reference in footnote 6) obtain $1-(3/2\pi)\,g^2=1-0.48\,g^2$. Our result reduces to theirs in the limit $\mu=0$.

§ 4. Discussion of Experiments and Conclusion

Data bearing on the nuclear scattering of π-mesons comes from experiments by Fretter,[15] Brown and McKay,[16] by Lovati et al,[17] and by Camerini et al,[18] to cite some of the more recent work. The first three experiments were performed with cloud chambers and the last with photographic plates. The cloud chamber experiments agree in their finding of rather small cross sections ($\lesssim 10^{-27}\,\mathrm{cm}^2$ per nucleon) for the large angle scattering of particles in penetrating showers (which include protons in addition to π-mesons). However, the cloud chamber experiments yield an interaction cross section (i.e. a cross section for the production of nuclear stars and secondary penetrating showers) for the penetrating particles an order of magnitude greater. Since an appreciable number of the nuclear stars produced by the secondary penetrating particles may be due to the recoil nucleons associated with π-meson scatterings,[19] the scattering cross section may actually be closer to the interaction cross section.

Some evidence supporting the interpretation of the interaction cross section as chiefly a scattering cross section is provided by the photographic experiment. Twenty-nine π-mesons (with energies between 85 and 1300 Mev.) are observed to produce nuclear stars in the emulsion; of these 29 stars, 7 are accompanied by single "shower" particles (in one case the "shower" particle was identified as a π-meson) whereas 22 are not accompanied by any "shower" particles. It is tempting to regard the 7 1_p stars examples of direct π-meson scattering and an appreciable fraction of the 22 0_p stars as examples of charge exchange scattering (with the π^0 decaying into two unobserved gamma rays). The interaction cross section, on the basis of rather poor statistics, was found to be close to geometrical. On the above interpretation, a rather large scattering cross section, say $2-5\cdot 10^{-26}\,\mathrm{cm}^2$ per nucleon, would be indicated.

In order to compare theory with experiment, it is necessary to know the meson coupling constants and the energies of the π-mesons. It seems most reasonable to evaluate the coupling constants from the experiment on the absorption of slow π-mesons by hydrogen.[20] It is found that:[21]

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15) W. B. Fretter, Phys. Rev. **76** (1949), 511.

16) W. W. Brown and A. S. McKay, Phys. Rev. **77** (1850), 342.

17) Lovati, Mura, Salvini and Tagliaferri, Nuovo Cim. VI (1949), 207.

18) Camerini, Fowler, Lock and Muirhead, Phil. Mag. **41** (1950), 413 and private communication.

19) The possibility of this interpretation depends on the fact that a fast meson can transfer a large fraction of its energy to a nucleon as a result of an elastic collision. Thus, according to (14), although a 5 Mev π-meson scattered through angles of 60°, 120°, 180°, will transfer 0.7, 1.8 and 2.31 Mev. respectively to a nucleon at rest, a 400 Mev. π-meson will transfer 113, 233 and 268 Mev. and a 1600 Mev. π-meson will transfer as much as 835, 1270 and 1356 Mev. at the respective angles of scattering.

20) W. Panofsky et al, Phys. Rev. in press; unfortunately, the resolution is only good enough to place an upper limit of 0.21 on $p_0/\mu_0 c$ (p_0 is the momentum of π^0, μ_0 its mass). The best

$S(S)$: $(g_P-g_N)^2/\hbar c=0.012$,

$PS(PS)$: $(g_P+g_N)^2/\hbar c=0.11$.

We take $g=(g_P-g_N)/\sqrt{2}$ for $S(S)$,

$g=(g_P+g_N)/\sqrt{2}$ for $PS(PS)$.

As far as the meson energy is concerned, we can use the numbers for $E_1=3\mu$ in Table I and II; the average energy of the π-mesons in the experiments cited can not be far from this value. We find for the direct scattering cross sections for the four theories (processes I and II are averaged for the purposes of this discussion):

$$\sigma_{S(S)}\cong 5\times 10^{-31}\ \text{cm}^2, \qquad \sigma_{S(V)}\cong 2\times 10^{-25}\ \text{cm}^2,$$

$$\sigma_{PS(PS)}\cong 3\times 10^{-30}\ \text{cm}^2, \qquad \sigma_{PS(PV)}\cong 2\times 10^{-25}\ \text{cm}^2.$$

The charge exchange cross section in all four theories is about twice the direct scattering cross section. In evaluating $\sigma_{S(S)}$ and $\sigma_{PS(PS)}$, we have neglected the higher order correction since $g_{S(S)}{}^2/\hbar c$ and $g_{PS(PS)}{}^2/\hbar c$ are so small (cf. eqs. (62) and (63)[22]). If we compare $\sigma_{S(S)}$ and $\sigma_{PS(PS)}$ with experiment assuming that the scattering cross section is comparable with the interaction cross section, it would appear that the theoretical values are much too low. On the other hand, the value of $\sigma_{S(V)}$ and $\sigma_{PS(PV)}$ are much too high. In view of the large values of $g_{S(V)}{}^2/\hbar c$ and $g_{PS(PV)}{}^2/\hbar c$, indicated by the π^- absorption experiment[20] the higher order corrections for the derivative couplings must be very important; furthermore. it is clear that the strong increase of scattering cross section with energy cannot be real and must be cut down by the reaction of the meson field. Since neither the reactive corrections nor the damping effects can really be calculated for the $S(V)$ and $PS(PV)$ theories,[23] we can only conclude that the derivative couplings

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value for the ratio of π^0-emission to γ-emission appears to be about 1.5 (W. Panofsky, private communication).

21) R. Marshak and A. Wightman, Phys. Rev.; in press: we choose the values which are found assuming identical field and coupling properties for both the charged and neutral π-mesons. The values given for $S(S)$ and $PS(PS)$ are to be regarded as lower limits since (f_0/μ_0) may be appreciably smaller. Similar calculation have been carried out by Aidzu, Fujimoto, Fukuda, Hayakawa, Takayanagi, Takeda and Yamaguchi (paper to appear shortly) for spin zero mesons. The same method applied to $PS(PV)$ leads to a value much larger than unity for $(g_P+g_N)^2/\hbar c$ and to an equally meaningless result for the case $S(V)$. Strictly speaking, the $S(V)$ neutral meson is excluded because the two gamma emission is forbidden (cf. J. Steinberger, Phys. Rev. **76** (1949), 1180). However, in order to make a numerical comparison for the scattering problem, we choose $(g_P+g_N)^2/\hbar c\approx 1$ in the $PS(PV)$ case and $(g_P-g_N)^2/\hbar c\approx 1$ in the $S(V)$ case.

22) We have found that the higher order corrections for $E\gg\mu$ but $\ll M$ are only slightly larger than in the Thomson limit $(E=\mu)$, (cf. also E. Corinaldesi and G. Field, Phil. Mag. **41** (1950), 364 referred to in footnote 6).

23) cf. W. Heitler, Proc. Camb. Phil. Soc. **37** (1941), 291; W. Heitler and H. W. Peng, Proc. Roy. Irish Acad. **49** (1943), 101A.

can easily lead to geometrical cross sections for the π-meson scattering.

It thus appears that the non-derivative coupling theories $S(S)$ and $PS(PS)$ are self-consistent weak coupling theories whose predictions can be directly compared with experiment. If our proposed identification of a major portion of the interaction cross section with the scattering cross section is confirmed by further experiment, the $S(S)$ and $PS(PS)$ theories could be discarded. On the other hand, the weak coupling predictions of the derivative coupling theories $S(V)$, $PS(PV)$[24] lead to contradictory and physically meaningless results. One qualitative prediction of all derivative coupling theories may perhaps be believed, namely the large increase in scattering cross section in the energy region $\mu \leq E \leq M$; this behavior is also found in the strong coupling theories[25] which are probably much more appropriate for derivative couplings. This is in striking contrast to the decreasing scattering cross section predicted by the non-derivative coupling theories.

Our results emphasize the importance of performing an experiment on the scattering of low energy π-mesons (i.e. those produced artificially by high energy accelerators) by protons in order to decide whether the cross section is an increasing or decreasing function of the energy, and to determine a good absolute value. Evidence from other sources, e.g. the photon production of π-mesons, favors a pseudoscalar field for the π-meson and the low energy experiment would help to decide whether the pseudovector coupling is indeed correct. It may even be possible to determine whether the strong coupling theory in its present form gives a suitable description of the meson scattering.

This work was supported by the joint program of the ONR and AEC.

Appendix

In this appendix most of the integrals occurring in the higher order scattering calculations will be evaluated. The methods developed in the appendix to Feynman's paper on the Space-Time Approach to Quantum Electrodynamics (reference 4) are most useful for this purpose and will be employed throughout.

(a) Nucleon Self Energy

It is convenient to notice that in the self energy integral (36) and in the subsequent integrals to be discussed

$$(k^2-\mu^2)^{-1}C(k^2-\mu^2)=(k^2-\mu^2)^{-1}-(k^2-\lambda^2)^{-1}$$
$$=-\int_{\mu^2}^{\lambda^2} dL(k^2-L)^{-2} \qquad (1a)$$

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[24] The spin one theories can be included in this class.

[25] W. Pauli, "*Meson Theory of Nuclear Forces*", Interscience Publishers, 1946.

where L is considered to have an infinitesimal negative imaginary part. This allows us to do convergent integrals on k directly, leaving at the end an integration on the parameter L from μ^2 to λ^2. The self-energy matrixes for the $PS(PS)$ and $S(S)$ theories thus have the form

$$\begin{aligned} PS(PS): &\quad g^2\gamma_5 W\gamma_5, \\ S(S): &\quad g^2 W \end{aligned} \tag{2a}$$

where

$$W=-\frac{i}{\pi}\int d^4k\,\frac{\boldsymbol{p}-\boldsymbol{k}+M}{(p-k)^2-M^2}\cdot\frac{dL}{(k^2-L)^2}. \tag{3a}$$

Using Feynman's integral (16a) from his appendix, and following his notation, the requisite integrals for W are

$$\begin{aligned} (8i)\int dL(1\,;\,k_o)d^4k(k^2-L)^{-2}(k^2-2p\cdot k)^{-1} \\ =\int_{\mu^2}^{\lambda^2}dL\int_0^1(1\,;\,(1-x)p_o)2x[M^2(1-x)^2+Lx]^{-1}dx \\ =2\int_0^1(1\,;\,(1-x)p_o)[2\log(\lambda/M)+\log x-\log((1-x)^2+\omega x)]dx. \end{aligned} \tag{4a}$$

The integrations in (4a) are elementary and yield for W

$$\begin{aligned} W=&-(1/8\pi)(\boldsymbol{p}+M)\{4\log(\lambda/M)+2-\omega\log\omega-2\sqrt{\tau}\,T(\omega)\} \\ &+(1/8\pi)\boldsymbol{p}\{2\log(\lambda/M)-\frac{1}{2}+\omega+\omega(1-\frac{1}{2}\omega)\log\omega-\omega\sqrt{\tau}\,T(\omega)\}. \end{aligned} \tag{5a}$$

Where $\bar{\omega}=(\mu/M)^2$ and $T(\omega)$ and τ are defined by (37)′. For the $PS(PS)$ theory we note that $\gamma_5(\boldsymbol{p}+M)\gamma_5=\boldsymbol{p}-M$ gives zero applied to a free nucleon wave function. For the $S(S)$ theory we may replace $\boldsymbol{p}$ by M in W after applying the matrix to a free nucleon wave function. The results for ΔM are given in (37) and (38).

(b) **Renormalization of g**

The factor r defined by (40)′ for the $PS(PS)$ theory may be abbreviated as

$$r_{PS}=\gamma_5 R\gamma_5 \tag{1b}$$

where

$$\begin{aligned} R=&-(i/\pi)\int(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\boldsymbol{p}M^{-1}(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}(k^2-L)^{-2}d^4k\,dL \\ =&-(i/\pi)\int\frac{\boldsymbol{p}-\boldsymbol{k}+M}{(k^2-2p\cdot k)}\cdot\frac{\boldsymbol{p}}{M}\cdot\frac{\boldsymbol{p}-\boldsymbol{k}+M}{(k^2-2p\cdot k)}\,\frac{d^4k\,dL}{(k^2-L)^2}. \end{aligned} \tag{2b}$$

This integral can be handled according to formula (20a) of Feynman's appendix.

We find

$$(8i)\ \int d^4k dL(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2p\cdot k)^{-2}=M^{-2}[\log\omega-2(\omega/\sqrt{\tau})T(\omega)], \tag{3b}$$

$$(8i)\ \int k_\sigma d^4k dL(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2p\cdot k)^{-2}$$
$$=M^{-2}p_\sigma[-2+\omega\log\omega+(2/\sqrt{\tau})\omega(2-\omega)T(\omega)], \tag{4b}$$

$$(8i)\ \int k_\sigma k_\nu d^4k dL(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2p\cdot k)^{-2}$$
$$=-M^{-2}p_\sigma p_\nu[1+2\omega+\omega(1-\omega)\log\omega-(2/\sqrt{\tau})\omega^2(3-\omega)T(\omega)]$$
$$+\frac{1}{2}\delta_{\sigma\nu}[2\log(\lambda/M)-1/2+\omega+\omega(1-\frac{1}{2}\omega)\log\omega-\omega\sqrt{\tau}\,T(\omega)]. \tag{5b}$$

Insertion of the integrals (3b), (4b), and (5b) into the formula for R and use of the properties or γ_5 in the pseudoscalar case, yields the expressions (40)″ and (41) for r_{PS} and r_S.

Let us consider next the vertex contribution to the charge renormalization. The integral in (42) of the text is

$$-(i/\pi)\int\frac{(\boldsymbol{p}_2-\boldsymbol{k}+M)\gamma_5(\boldsymbol{p}_1-\boldsymbol{k}+M)\gamma_5 d^4k dL}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-L]^2} \tag{6b}$$

where we have used the abbreviations $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$, $\boldsymbol{p}_1=\boldsymbol{p}$. Since we are dealing with free nucleons we have $\boldsymbol{p}_1^2=\boldsymbol{p}_2^2=M^2$. Note that $\boldsymbol{p}_2^2=p_1^2+2p_1\cdot q+\boldsymbol{q}^2$ and hence $2p_1\cdot q=-\boldsymbol{q}^2$. The numerator of the integrand in (6b) may be simplified as follows: $(\boldsymbol{p}_2-\boldsymbol{k}+M)\gamma_5(\boldsymbol{p}_1-\boldsymbol{k}+M)\gamma_5=(\boldsymbol{p}_2-\boldsymbol{k}+M)(\boldsymbol{p}_1-\boldsymbol{k}-M)=-(\boldsymbol{p}_2-\boldsymbol{k}+M)\boldsymbol{k}$ where $\boldsymbol{p}_1-M$ on the right has been replaced by zero since it operates on a free nucleon spinor u_1 for momentum $\boldsymbol{p}_1$. From (42) we see that with an extra γ_5 on the left, it is necessary to form $-\bar{u}_2\gamma_5(\boldsymbol{p}_2-\boldsymbol{k}+M)\boldsymbol{k}=\bar{u}_2\gamma_5\boldsymbol{k}^2$. The numerator in (6b) is therefore equivalent simply to $\boldsymbol{k}^2$ as far as the matrix element between $\bar{u}_2$ and u_1 is concerned. The k integration is best performed after combining the three terms in the denominator of the integrand in (6b) using Feynman's identity

$$\frac{1}{a^2bc}=6\int_0^1 dt\int_0^{1-t}\frac{xdx}{[ax+b(1-x-t)+ct]^4}. \tag{7b}$$

For brevity we introduce the notation

$$Av f(x,t)\equiv 6\int_0^1 dt\int_0^{1-t}xdx f(x,t) \tag{8b}$$

where $f(x,t)$ is an arbitrary function; the integral in (8b) is a kind of average of the function $f(x,t)$. With the technique of (7b), the integral (6b) becomes

$$(6b)=-(i/\pi)\int dL Av\int d^4k\boldsymbol{k}^2[(\boldsymbol{k}-Q)^2-\Delta]^{-4}$$

$$=-(i/\pi)\int dLAv\int d^4k(\boldsymbol{k}^2+2k\cdot Q+Q^2)[\boldsymbol{k}^2-\Delta]^{-4} \tag{9b}$$

$$=(1/\pi)\int dLAv\{-(12\Delta)^{-1}+\boldsymbol{Q}^2(24\Delta^2)^{-1}\}$$

where $\boldsymbol{Q}=(1-x-t)\boldsymbol{p}_1+t\boldsymbol{p}_2=(1-x)\boldsymbol{p}_1+t\boldsymbol{q}$ and $\Delta=xL+\boldsymbol{Q}^2$. Note that $\boldsymbol{Q}^2=M^2(1-x)^2+\boldsymbol{q}^2[t^2-t(1-x)]$ thus establishing the fact that (6b) is a function of $\boldsymbol{q}^2$ alone, as indicated in (43) of the text. In obtaining (9b) we have used the results $\int d^4k[\boldsymbol{k}^2-\Delta\ ^{-4}=(i/24)\Delta^{-2}$ and $\int \boldsymbol{k}^2d^4k[\boldsymbol{k}^2-\Delta]^{-4}=-(i/12)\Delta^{-1}$. The integral $\int k\cdot Qd^4k[\boldsymbol{k}^2-\Delta]^{-4}$ clearly gives zero.

According to the physical argument following (43), it is sufficient to obtain (6b) for the special case $\boldsymbol{q}^2=\mu^2$. After performing the L integration in (9b), and remembering that $\lambda\gg M$, we find for (6b)

$$8\pi v_{PS}(\mu^2)=-4\log(\lambda/M)+3+4\int_0^1 dt\int_0^{1-t}dx\log[(1-x)^2+\omega x+\omega(t^2-t(1-x))]$$
$$+2\int_0^1 dt\int_0^{1-t}dx\frac{(1-x)^2+\omega(t^2-t(1-x))}{(1-x)^2+x+(t^2-t(1-x))}.$$

Inverting the order of integration on x and t gives the reduction

$$8\pi v_{PS}(\mu^2)=-4\log(\lambda/M)+3+4\int_0^1 dx(1-x)\log\{(1-x)^2+\omega x\}$$
$$-8\int_0^1 dx\{1-x-2\Phi\tan^{-1}[(1-x)(2\Phi)^{-1}]\} \tag{10b}$$
$$+1-4\int_0^1 dx\{x\Phi^{-1}\tan^{-1}[(1-x)(2\Phi)^{-1}]\}$$

where $\Phi^2(x)=(1/\omega-1/4)(1-x)^2+x$.
A similar argument for the $S(S)$ case yields

$$8\pi v_S(\mu^2)=8\pi v_{PS}(\mu^2)+16\omega^{-1}\int_0^1 dx\, x\Phi^{-1}\tan^{-1}[(1-x)(2\Phi)^{-1}]. \tag{11b}$$

The integrations in (10b) and (11b) were in part performed numerically for $\omega^{-1}=(6.4)^2=40.96$

(c) **Bubble Terms**

The only difference between the operator 0_I defined in (49)′ and the self energy operator W of (3a) lies in the fact that $(\boldsymbol{p}_1+\boldsymbol{k}_1)^2\neq M^2$. Introducing the notation $\boldsymbol{q}\equiv\boldsymbol{p}_1+\boldsymbol{k}_1$ and $a\equiv\boldsymbol{q}^2/M^2$, and using Feynman's integral (16a), the integrals required for 0_I are

$$(8i)\int dL(1\,;\,k_0)d^4k(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2q\cdot k-M^2+\boldsymbol{q}^2)^{-1}=$$
$$=2\int_0^1(1\,;\,yq_0)dy[2\log(\lambda/M)+\log(1-y)-\log\{ay^2-(a+\omega-1)y+\omega\}]. \tag{1c}$$

It is important to notice that the quadratic expression $ay^2-(a+\omega-1)y+\omega$ vanishes for two values of y in the interval (0,1). This is a consequence of the fact that $\boldsymbol{q}^2 \geq (M+\mu)^2$, with equality holding only in the Thomson limit. We denote the two roots for y by y_0 and y_1, requiring $y_0 \leq y_1$. In the Thomson limit the roots become equal: $y_0=y_1=\mu(M+\mu)^{-1}$. The presence of these roots offers no difficulty when it is recalled from (1a) that the parameter L is to be regarded as having an infinitesimal negative imaginary part.

Carrying through the integration in (1c) gives

$$(8i)\int dLd^4k(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2q\cdot k-M^2+\boldsymbol{q}^2)^{-1}=4\log(\lambda/M)+U, \tag{2c}$$

$$U=-2\log(\boldsymbol{q}^2M^{-2})+2+2i\pi(y_1-y_0)-2y_0\log y_0-2(1-y_0)\log(1-y_0)$$
$$-2y_1\log y_1-2(1-y_1)\log(1-y_1), \tag{3c}$$

$$(8i)\int dLk_0d^4k(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2q\cdot k-M^2+\boldsymbol{q}^2)^{-1}=q_0(2\log(\lambda/M)+X), \tag{4c}$$

$$X=-\log(\boldsymbol{q}^2M^{-2})-\frac{1}{2}+i\pi(y_1^2-y_0^2)+y_0-y_0^2\log y_0-(1-y_0^2)\log(1-y_0)$$
$$+y_1-y_1^2\log y_1-(1-y_1^2)\log(1-y_1). \tag{5c}$$

Assembling the parts of 0_I from the integral

$$0_I=-\frac{i}{\pi}\int\frac{\boldsymbol{p}_1+\boldsymbol{k}_1-\boldsymbol{k}+M}{(\boldsymbol{p}_1+\boldsymbol{k}_1-\boldsymbol{k})^2-M^2}\cdot\frac{d^4kdL}{(\boldsymbol{k}^2-L)^2} \tag{6c}$$

gives the result (50) quoted in the text.

We may briefly indicate here what happens when the argument of the text, applying the charge and mass renormalization procedure to the bubble term, is carried out for the $S(S)$ theory. The bubble matrix element is obtainable from (49) by replacing all γ_5's by unity. Rearranging 0_I conveniently as

$$0_I=-(1/8\pi)\{(\boldsymbol{p}_1+\boldsymbol{k}_1-M)(2\log(\lambda/M)+U-X)+M(6\log(\lambda/M)+2U-X)\}, \tag{7c}$$

the charge renormalization, taken in conjunction with the first term of (7c), requires consideration of the quantity

$$-(8\pi)^{-1}(2\log(\lambda/M)+U-X)+r_S \tag{8c}$$

analogous to (53). The quantity (8c) is finite according to (41) and (40)″. Similarly the mass renormalization, taken together with the second term of (7c) leads to a net correction to

$$-(8\pi)^{-1}M(6\log(\lambda/M)+2U-X)-\Delta M_S. \tag{9c}$$

Eq. (38) for ΔM_S shows that (9c) is also finite, independent of cut-off.

The bubble diagram B_{II} of Fig. 2 presents us with an operator 0_{II} obtainable from 0_I in (49)′ through the replacement $\boldsymbol{p}_1+\boldsymbol{k}_1 \rightarrow \boldsymbol{p}_1-\boldsymbol{k}_2$. With $a \equiv M^{-2}(\boldsymbol{p}_1-\boldsymbol{k}_2)^2$, the quadratic expression $ay^2-(a+\omega-1)y+\omega$ appearing in (1c) is readily shown to have two negative real roots $-y_0$ and $-y_1$ which coalesce to $-\mu(M-\mu)^{-1}$ in the Thomson limit. The logarithm in (1c) is therefore always well defined. The quantities analogous to U and X of (3c) and (5c) are

$$U=-2\log\left((p_1-k_2)^2M^{-2}\right)+2-2(1+y_0)\log(1+y_0)+2y_0\log y_0$$
$$-2(1+y_1)\log(1+y_1)+2y_1\log y_1, \qquad (10c)$$

$$X=-\log\left((p_1-k_2)^2M^{-2}\right)-\frac{1}{2}-y_0-y_0^2\log y_0-(1-y_0^2)\log(1+y_0)$$
$$-y_1-y_1^2\log y_1-(1-y_1^2)\log(1+y_1) \qquad (11c)$$

where, of course, the y_0 and y_1 have a different meaning than in (3c) and (5c).

(d) **Vertex Terms**

The characteristic integral Λ_I defined by (56)′ is almost the same as the integral (6b) for the vertex correction to the scattering of a nucleon by a potential; the difference comes from the fact that $(p_1+k_1)^2\neq M^2$. With the abbreviation $q=p_1+k_1$,

$$\Lambda_I=-\frac{i}{\pi}\int\frac{(q-k+M)\gamma_5(p_1-k+M)\gamma_5 d^4k\,dL}{[(q-k)^2-M^2][(p_1-k)^2-M^2][k^2-L]^3}. \qquad (1d)$$

According to (7b) and the definition of Av in (8b), we may write

$$\Lambda_I=-\frac{i}{\pi}\int dL\,Av\int\frac{(q-k+M)\gamma_5(p_1-k+M)\gamma_5 d^4k}{[(k-Q)^2-\Delta]^4}$$
$$=-\frac{i}{\pi}\int dL\,Av\int\frac{(q-k-Q+M)\gamma_5(p_1-k-Q+M)\gamma_5 d^4k}{[k^2-\Delta]^4} \qquad (2d)$$

where $Q=(1-x-t)p_1+tq$ and $\Delta=Q^2+Lx+(M^2-q^2)t$. Only the even powers of k contribute to the second form of the integral in (2d). Using the fact that Λ_I is to be applied to the spinor u_1 obeying $p_1u_1=Mu_1$ we obtain

$$\Lambda_I=\pi^{-1}\int dL\,Av\{-(12\Delta)^{-1}-(q-Q+M)Q(24\Delta^2)^{-1}, \qquad (3d)$$

After the L integration is performed

$$8\pi\Lambda_I=-4\log(\lambda/M)+3+4\int_0^1dt\int_0^{1-t}dx\log\left[M^{-2}(Q^2+\mu^2x+(M^2-q^2)t)\right]$$
$$-2\int_0^1dt\int_0^{1-t}dx\frac{(q-Q+M)Q}{Q^2+\mu^2x+(M^2-q^2)t}. \qquad (4d)$$

In the Thomson limit the x and t integrations are not difficult. All energy-momentum four-vectors reduce to their fourth components alone and all matrices γ_4 may be replaced by unity (after the γ_5's have been eliminated). A change to the fourth component of Q as independent variable, in place of t, is found to be convenient since the x integration may then be performed immediately. The remaining integration on Q_4 is possible analytically but simpler to do numerically. We may note that in the Thomson limit the diagram V_I' of Fig. 2 contributes the same amount as V_I. The contribution of V_{II} is obtainable from (4d) by making the replacement $\boldsymbol{q}=\boldsymbol{p}_1-\boldsymbol{k}_2$.

It should be remarked that in (4d), for $x=1-t$ and $\boldsymbol{q}=\boldsymbol{p}_1+\boldsymbol{k}_1$ the expression $\Delta=\boldsymbol{Q}^2+\mu^2(1-t)+(M^2-\boldsymbol{q}^2)t=\boldsymbol{q}^2t^2-(\boldsymbol{q}^2+\mu^2-M^2)t+\mu^2$ has two roots for t in the interval (0,1) just as for the bubble integral (1c). This causes no difficulty, however, and may be handled as in the bubble case.

The scalar theory gives, in place of (3d),

$$\Lambda_I=\pi^{-1}\int dL Av\{-(12\Delta)^{-1}+(\boldsymbol{q}-\boldsymbol{Q}+M)(2M-\boldsymbol{Q})(24\Delta^2)^{-1}\}. \tag{5d}$$

Replacement of $\boldsymbol{q}$ by $\boldsymbol{p}_1-\boldsymbol{k}_2$ gives Λ_{II}.

(e) Convergent Terms

The diagram C_I of Fig. 2 requires the integral (59) ($PS(PS)$ theory).

$$C_I=\frac{1}{\pi}\int\frac{\boldsymbol{k}(\boldsymbol{q}-\boldsymbol{k}+M)\boldsymbol{k}d^4k}{[\boldsymbol{k}^2-2p_2\cdot k][(\boldsymbol{q}-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-2p_1\cdot k][\boldsymbol{k}^2-\mu^2]} \tag{1e}$$

The matrix element of (1e) between $\bar{u}_2$ and u_1 is understood. For handling such an integral with a product of four terms in the denominator, we use the identity

$$(a\,b\,c\,d)^{-1}=6\int_0^1 dt\int_0^{1-t}dx\int_0^{1-t-x}dy\,[at+bx+cy+d(1-t-x-y)]^{-4} \tag{2e}$$

and the notation

$$Av\,f(t,x,y)\equiv 6\int_0^1 dt\int_0^{1-t}dx\int_0^{1-t-x}dy\,f(t,x,y). \tag{3e}$$

Setting $a=(\boldsymbol{q}-\boldsymbol{k})^2-M^2,\ b=\boldsymbol{k}^2-\mu^2,\ c=\boldsymbol{k}^2-2p_1\cdot k,\ d=\boldsymbol{k}^2-2p_2\cdot k,$

$$C_I=\pi^{-1}Av\int d^4k[(\boldsymbol{k}-\boldsymbol{Q})^2-\Delta]^{-4}\boldsymbol{k}(\boldsymbol{q}-\boldsymbol{k}+M)\boldsymbol{k}, \tag{4e}$$

$$\boldsymbol{Q}=t\boldsymbol{q}+y\boldsymbol{p}_1+(1-t-x-y)\boldsymbol{p}_2, \tag{5e}$$

$$\Delta=\boldsymbol{Q}^2-(\boldsymbol{q}^2-M^2)t+\mu^2x. \tag{6e}$$

A shift of variable in (4e) from $\boldsymbol{k}$ to $\boldsymbol{k}+\boldsymbol{Q}$, and performance of the k integrations gives

$$\begin{aligned}C_I&=\pi^{-1}Av\int d^4k[\boldsymbol{k}^2-\Delta]^{-4}(\boldsymbol{k}+\boldsymbol{Q})(\boldsymbol{q}-\boldsymbol{k}-\boldsymbol{Q}+M)(\boldsymbol{k}+\boldsymbol{Q})\\&=i(24\pi)^{-1}Av\{(\boldsymbol{q}+3\boldsymbol{Q}-2M)\Delta^{-1}+\boldsymbol{Q}(q-\boldsymbol{Q}+M)\boldsymbol{Q}\Delta^{-2}\}.\end{aligned} \tag{7e}$$

The integral C_{II} appropriate for the diagram C_{II} may be obtained from (7e) if $\boldsymbol{q}$ is interpreted to mean $\boldsymbol{p}_1-\boldsymbol{k}_2$.

We have not succeeded in carrying out the parametric integrations in (7e) for the general case. In the Thomson limit, however, the integration is readily performed since $\boldsymbol{Q}$ and Δ become independent of the parameter y. As is seen from (5e), in the limit $\boldsymbol{q}=(M+\mu)\gamma_4$, $\boldsymbol{p}_1=\boldsymbol{p}_2=M\gamma_4$, $\boldsymbol{Q}$ reduces to $\boldsymbol{Q}=[M(1-x)+\mu t]\gamma_4$. A further simplification is achieved by changing to the fourth component of $\boldsymbol{Q}$ as independent variable in place of t, since this makes Δ linear in x. Several of the integrals in (7e) reduce, in the Thomson limit, to integrals involving Δ^{-1} already performed for the vertex term (4d). Somewhat more difficult are the integrals containing Δ^{-2}. After the x integration there is left an integration over t (or Q) of Δ^{-1} with x replaced by $1-t$. From (6e), however, for $x=1-t$ and the Thomson limit, $\Delta=(Q-\mu)^2$. The double root of Δ at $Q=\mu$ is the transform of the double root in (1c) encountered for the quadratic argument of the logarithm. There is thus an apparent divergence in the integrals $\int\cdots(Q-\mu)^{-2}dQ$ which must be performed over the interval $0\leq Q\leq M+\mu$. The situation is remedied by noting that if the Thomson limit is not exactly reached, the degeneracy in the roots of Δ is removed. It is then possible to take the principal value of the integrals and subsequently allow the roots to approach one another. Such a situation does not arise for the integral C_{II} where we find instead integrals like $\int\cdots(Q+\mu)^{-2}dQ$ over the interval $0\leq Q\leq M-\mu$.

For the $S(S)$ theory, corresponding to (7e),

$$C_I=i(24\pi)^{-1}Av\{(\boldsymbol{q}+3\boldsymbol{Q}-10M)\Delta^{-1}+(\boldsymbol{Q}-2M)(\boldsymbol{q}-\boldsymbol{Q}+M)(\boldsymbol{Q}-2M)\Delta^{-2}\} \tag{8e}$$

where $\boldsymbol{q}=\boldsymbol{p}_1+\boldsymbol{k}_1$. Diagram C_{II} gives a similar result with $\boldsymbol{q}=\boldsymbol{p}_1-\boldsymbol{k}_2$.

Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On the Production of Cosmic Ray Mesons

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(Received July 31, 1950)

§ 1. Introduction

Recent developments in experiments with artificial mesons have yielded much information regarding the nature of the π-meson, but its quantitative properties are not yet obvious, mainly because of difficulties with meson theory. The existence at present of more experimental data, especially the comprehensive analyses of cosmic ray phenomena at high energy, urges us to draw up a consistent picture of the meson. Unfortunately, however, there may be some ambiguity and inconsistency among the various experiments, a part of which will be eliminated by adequate corrections accounting for experimental conditions. Such experimental results, though lacking in perfection, should first be analyzed in the light of current meson theories since the discrepancy between experiment and theory will give a clue to later developments. In the present paper we attempt to give a brief summary of mesonic interactions in cosmic rays together with the future aspects of this field.

First of all we give some kinematical relations between an observed coordinate system and the center of mass system of two colliding nucleons with few assumptions about the mechanism of the meson production (§ 2). Next a brief account of our theory,[1] to which we tentatively refer, and its characteristic features are presented (§ 3). The greater part is devoted to the analysis and interpretation of several remarkable points of nucleonic showers (§ 4).

§ 2. Lorentz Transformation Between Laboratory and Center of Mass Systems

For the interpretation of meson showers produced by a nucleon-nucleon collision, it is necessary to provide the Lorentz transformation between the laboratory system (l. s.) and the center of mass system (c. m. s.), since it is more convenient to work in the latter system.[2),3),4),5),6),7)]

We label the quantities in c. m. s. and l. s. with and without asterisks, respectively. Suppose a nucleon with energy E and momentum P collides with a nucleon at rest of mass M. Then the Lorentz factor ξ, transforming one system into the other system, is given by

$$\xi=\sqrt{(E+M)/2M}=E^*/M, \tag{2.1}$$

where we set the velocity of light to unity.

The transformation formulae of momentum p and angle θ of a produced meson are expressed as

$$p^*=\{(\epsilon-\beta p\cos\theta)^2-(1-\beta^2)\mu^2\}^{1/2}\xi, \tag{2.2}$$

$$\tan\theta^*=p\sin\theta/(p\cos\theta-\beta\epsilon)\xi, \tag{2.3}$$

or

$$p=\{(\epsilon^*+\beta p^*\cos\theta^*)-(1-\beta^2)\mu^2\}^{1/2}\xi, \tag{2.2'}$$

$$\tan\theta=p^*\sin\theta^*/(p^*\cos\theta^*+\beta\epsilon^*)\xi, \tag{2.3'}$$

where ϵ and μ represent the energy and mass of the meson and β is the relative velocity between both systems, $\xi=1/\sqrt{1-\beta^2}$.

The momentum and angular distribution in l. s. $F(p,\theta)\,dpd\theta$ and that in c.m.s $f(p^*,\theta^*)\,dp^*d\theta^*$ are connected by

$$F(p,\theta)\,dpd\theta=f(p^*,\theta^*)Jdpd\theta, \tag{2.4}$$

where J represents the Jacobian between both sets of variables:

$$J=p(\epsilon-\beta p\cos\theta)/\epsilon p^*. \tag{2.5}$$

In the case of an isotropic angular distribution in c.m.s., we obtain

$$F(p,\theta)=g(p^*)p^2(\epsilon-\beta p\cos\theta)\sin\theta/2\epsilon p^{*2}, \tag{2.6}$$

where $g(p^*)$ is the momentum distribution in c.m.s. The main behavior of the angular distribution is governed by p^* in the denominator of (2.6). For small θ and large p, the emitted angle of a meson in l.s. is restricted as follows

$$(\mu/p)+(1/\xi)\gtrsim\theta\gtrsim|(\mu/p)-(1/\xi)|, \tag{2.7}$$

which means that the emitted mesons are restricted to a cone with the vertical angle $\theta\sim 1/\xi$ for $p>\xi\mu$.

The angular or the momentum distribution is obtained by integrating $F(p,\theta)$ over p or θ, provided the function $g(p^*)$ is known. The simplest case is $g(p^*)=\delta(p^*-p_0^*)$, namely, the unique momentum p_0^* in c.m.s., as adopted by Rossi[6] and Gamba and Radicati.[4] The angular distribution is expressed as

$$\Phi(\theta)=\frac{\sin\theta\{\beta\epsilon_0^*\cos\theta+(\epsilon_0^*-\xi^2\beta^2\mu^2\sin^2\theta)^{1/2}\}^2}{2\xi^2(1-\beta^2\cos^2\theta)(p_0^{*2}-\xi^2\beta^2\mu^2\sin^2\theta)^{1/2}}, \tag{2.8}$$

where it should be noted that the angle of the emitted meson is restricted so as to have $\sin\theta\lesssim p_0^*/\xi\beta\mu$. This maximum angle corresponds to the meson emitted at an angle in the backward hemisphere. As inferred from (7), $\Phi(\theta)/\theta$ increases only slowly for $\theta\lesssim 1/\xi$. The momentum distribution is

$$\Psi(p)=\{\xi^2\beta^2p^2-(\xi\epsilon-\epsilon_0{}^*)^2\}^{1/2}\epsilon_0{}^*/\xi^2\beta p_0{}^*\epsilon. \tag{2.9}$$

For larger p, $\Psi(p)$ has a maximum at $\xi\mu^2/\epsilon_0{}^*$ and p is restricted such that $2\xi(\epsilon_0{}^*-\mu)\gtrsim p\gtrsim\xi\mu^2/2(\epsilon_0{}^*-\mu)$. For smaller p, $\Psi(p)$ tends to $\propto p^{-1/2}$.

If $g(p^*)$ is represented by a power of p^* as assumed by Heisenberg,[3] $g(p^*)\propto(\mu/p^*)^n$, $\Phi(\theta)$ and $\Psi(p)$ are obtained as follows. Approximating p^* as $\theta-\beta p\cos\theta\approx(1-\beta\cos\theta)p+\mu^2/2p$, the integration is carried out for two ranges of p, $p\geq\bar{p}$ and $p\leq\bar{p}$, where $\bar{p}=\mu/\sqrt{2(1-\beta\cos\theta)}$. For $p>\bar{p}$, the upper limit of the integration must be put at p_0, the maximum acceptable momentum, but this does little to affect the result for $n\geq1$. Then we obtain

$$\Phi(\theta)\propto\begin{cases}\sin\theta/(1-\beta\cos\theta)^{\frac{n}{2}+2} & \text{for } p\geq\bar{p}, \quad (2.10a)\\ (1-\beta\cos\theta)^{\frac{n}{2}+1}\sin\theta & \text{for } p\leq\bar{p}. \quad (2.10b)\end{cases}$$

In this case also, the emitted mesons have a core with angular departure $\theta\sim1/\xi$. The momentum distribution is obtained as

$$\Psi(p)\propto\frac{p}{\beta\epsilon}\,\frac{\mu^n}{\{(\epsilon-\beta p)^2-\mu^2/\xi^2\}^{n/2}}\sim\begin{cases}(\mu/p)^{\frac{n}{2}}\xi^n & \text{for } p>\xi\mu,\\ (p/\mu)^{n/2} & \text{for } p<\xi\mu.\end{cases} \tag{2.11}$$

Thus $\Psi(p)$ has a maximum at $p=\xi\mu$.

For $n=0$, both $\Phi(\theta)$ and $\Psi(p)$ show logarithmic dependences on $(1-\beta\cos\theta)$ and p/μ, respectively, but are not shown here.

Generally speaking, almost all secondary mesons are restricted within the angle $\theta\leq1/\xi$ and their momenta concentrate about $p\sim\xi\mu$. Such properties are due to isotropic emission in c.m.s. If the emission is not isotropic, the results are more complicated. For example, there may exist double cones, if forward and backward emission are favored.[6],[7]

§ 3. Theory of Meson Production at High Energy

The cross-section for meson production in a nucleon-nucleon collision increases with increasing energy of the colliding nucleon, if the coupling between meson and nucleon contains a derivative of the wave function of the meson. It becomes larger than the area of the effective meson field, namely the geometrical cross-section, as long as we apply straightforward perturbation theory. This fact is hardly understandable and suggests that the higher order process is more effective than the lower one. Actually we can observe the multiple production of mesons at high energies. This experimental fact favors the existence of the coupling with derivative, as far as current meson theory is concerned. Therefore, we may ignore scalar meson theory and have only to consider the pseudovector coupling of pseudoscalar meson. We do not consider the spin 1 meson because it is highly improbable on the basis of various arguments.

The multiple production of mesons is extensively treated by Lewis et al. on semi-quantum mechanical grounds.[8] Their theory is extended by Fukuda and Takeda,[1] accounting for the charge exchange effect, which we refer hereafter as *I*. Their theories are essentially classical, since they take only the terms remaining as $\hbar \to 0$ and neglect the radiative correction. However, the classical treatment will be a good approximation, because the case where a considerable number of mesons is produced corresponds to the higher quantum state.

The probability for emitting n mesons is given in *I* as

$$\sigma(n)=\sigma_1 A(n), \tag{3.1}$$

$$A(n)=\left(\frac{g}{\pi}\right)^{2n}\frac{2^{2n}(n+2)}{3n!\,(2n)\,!}\left(\frac{w^*}{\mu}\right)^{2n} \tag{3.2}$$

where σ_1 is the scattering cross-section of a nucleon, g the pseudovector coupling constant and w^* is the energy transferred to all mesons in c.m.s. The latter quantity can not be determined by the present theory and will include some unknown effect, such as damping. Here we may tentatively assume that w^* is proportional to the maximum available energy in a collision

$$w^*=\eta 2(E^*-M)\equiv\eta W^*. \tag{3.3}$$

Whether η is a constant or not will be decided by comparison with experiments.

Because of the unsatisfactory nature of our theory, the absolute value of the cross-section is not convincing, so that we mainly account for the relative probability for emitting n mesons· Comparing the probabilities for n and $n+1$, we obtain

$$R(n)=\frac{A(n+1)}{A(n)}=\frac{g^2}{\pi^2}\frac{2(n+3)}{(n+1)^2(n+2)(2n+1)}\left(\frac{w^*}{\mu}\right)^2, \tag{3.4}$$

which is approximated for high multiplicity by

$$R(n)\sim(g^2/\pi^2 n^3)(w^*/\mu)^2. \tag{3.5}$$

The most probable multiplicity and the root mean square deviation are found to be

$$\bar{n}=(g^2/\pi^2)^{1/3}(w^*/\mu)^{2/3}, \tag{3.6}$$

$$\Delta n=\sqrt{\bar{n}/3}\ \cdot \tag{3.7}$$

These results may also express the classical nature of our theory, if one compares our results with the theory of Fermi,[9] who considers only the density of final state and ignores the interaction between nucleons and mesons. Both theories show an approximate coincidence, especially in the energy dependence.

These theories are seen to contradict the simplest assumption, that the energy of a produced meson is independent of the primary energy, as adopted by several authors.[4],[5],[6] The average energy of emitted mesons is

$$\bar{\epsilon}^*=w^*/\bar{n}=(\pi/g)^{3/2}w^{*1/3}\mu^{2/3}, \tag{3.8}$$

which increases slowly with incident energy. Which of the assumptions is better will be decided by comparison with experiments.

Our theory can further predict more physical quantities. The angular distribution of emitted mesons is nearly isotropic. As for final nucleons, they are concentrated forward and backward with the angle about μ/E^* if the scattering potential is vector or pseudovector type, whereas they are isotropic in other cases. The non-isotropic emission of the final nucleons may affect the angular distribution of mesons, and it may be possible that the fastest meson is emitted like the nucleons.

The charge dependence of produced meson is considerably dependent upon the type of scattering potential. The ratio of the numbers of neutral to charged mesons is estimated as 3/5 or 1/4 for the momentum of a nucleon $p^*=2M$ according to the symmetrical or the mixture of half symmetrical and half neutral potential. If we adopt the former case, we can obtain the total number of mesons by multiplying 1.36 by the number of charged mesons, N, in higher multiplicity.

$$n=1.36\ N, \qquad (3.9)$$

§ 4. Analysis of Experiments

Various experimental data concerning our problem show seeming discrepancies with each other, when we deal with the raw data. Such discrepancies mainly come from the different experimental methods and mostly disappear with an appropriate correction as shown in what follows. The most convincing experimental results thus obtained are discussed in the light of the above theory.

1. *The energy of the agent nucleons.* The kinetic energy of nucleons effective to produce mesons seems to be greater than about 2 Bev. These are referred to as the A-component in our previous paper.[10] Some method to assign the energy described in II are not always applicable to such high energies discussed here. We confine ourselves to the following two methods.

One is the absolute intensity of the associated shower phenomena, which is closely related to the energy of the primary rays through their energy spectrum. Unfortunately, however, we have few cases where the intensity is estimated. In only two available cases, we can estimate the agent energies as $\gtrsim$ 10 Bev and $\gtrsim$20 Bev in the experiments of Fretter,[11] and Janossy and Rochester,[12] respectively. The latter experimenral arrangement is used by Butler et al.,[13] who estimated the average agent energy as $\sim$7 Bev. But this seems to be an underestimation, since the whole incident energy is assumed to be transferred to the penetrating particles emitted within one steradian.[14] If we assume that half of the available energy is converted into mesons, $\eta=0.5$ in (3.3), the transferred energy in one collision is about 7 Bev for 20 Bev incident energy. Actually, on the one hand, there may be plural collisions in a lead nucleus and, on the other hand, a considerable part of the released energy may escape observation so that our estimation may not be inconsistent with *BRB*.

The other method is to make use of the integral multiplicity-frequency relation. This relation is well approximated to $\sim N^{-2}$ in the experiments at lower altitude if N is smaller than ten. In Fretter's result, this relation is fairly distorted as discussed later. If we correct this distortion, N^{-2} relation is found valid $N \geq 6$ and the steep behavior for smaller N is explained as due to the selection by the triggering device. This leads to the agent energy $\gtrsim 9$ Bev, provided energy is equal to about 1.5 N as estimated in *II*. The most remarkable evidence for this assumption is obtained by *FLO*, who find a marked change in the slope of the multiplicity spectrum at $N=6$, which is attributed to the magnetic cut-off at about 8 Bev.[6]

2. *Momentum distribution of mesons.* We have not yet enough data about the momentum distribution for individual nuclear collisions, but have the statistical data. At higher energies, the cross-section for meson production is inferred as a function of the ratio of the momenta of meson and nucleon from the similarity between the energy distributions for both components.[15],[16] The higher average energy of secondary mesons in *BRB* may be due to the absence of lower energy primaries. Their average momentum, ∼0.8 Bev/c, is too high provided the power distribution in c.m.s. is assumed, (2.11), but consistent with the unique energy assumption, (2.9). Nevertheless, the experimental evidence seems to be not so certain as to decide in favor of either theory.

The momentum distribution obtained by the Bristol group[17] shows a surprising coincidence with that obtained by the indirect analysis of Sands,[18] although the energy spectra of primary nucleons are somewhat different for both experiments because of the different altitudes. This will result from the lower efficiency of meson production by the lower energy primaries, where the energy spectra will differ.

3. *Charge dependence.* Also, for the charge of produced mesons, we have little direct data. We are obliged to argue this problem from indirect knowledge.

The ratio of charged to neutral mesons can be estimated from the total energy of electronic rays, $W^{(e)}$, and μ-mesons, $W^{(m)}$, referring to the following decay schemes:

neutral π-meson → two photons,

charged π-meson → charged μ-meson + neutrino,

charged μ-meson → electron (positron) + two neutrinos.

The total energies of both components are given by Rossi[19] as

$$W^{(e)}=0.285 \text{ Bev cm}^{-2}\text{ sec}^{-1},$$

$$W^{(m)}=0.289 \text{ Bev cm}^{-2}\text{ sec}^{-1}.$$

From the above, we can estimate the total energies transferred to the charged and neutral π-mesons, W_c and W_n, as

$$W_c=0.37\ \text{Bev cm}^{-2}\ \text{sec}^{-1},$$

$$W_n=0.22\ \text{Bev cm}^{-2}\ \text{sec}^{-1}.$$

These figures are consistent with the result of the symmetrical theory, which gives neutral-charged ratio 0.60 for $n=1$ and 0.36 for larger n.

The ratio of positive to negative mesons is 2 : 1 for a proton incident in both cases of coupling mentioned in § 3. If the main contribution to the positive excess comes from the mesons produced in the upper atmosphere,[20] where the primary nucleons consist of 1/3 neutrons and 2/3 protons,[21] and the mesons are mostly produced singly as argued lately, we get a positive excess of about 20%, well in agreement with observation.

4. *Angular distribution.* The angular distribution is often used to assign the energy of an incident ray, but it seems to be inappropriate considering the different results by the different experimental methods. The average angles of forward secondaries obtained by some authors are compared in Table I, with the primary energies assigned by the simple relation $\theta=1/\xi$, where the angle is corrected to the true one if the projected angle is measured.

Table I

			FLO				
Observer	*B*	light	heavy	total	*F*	*BM*[22]	*W*[23]
Average angle	30°	22°	30°	28°	26°	38°	14°
Primary energy (Bev)	5.8	12	5.8	7.0	8.1	3.3	30

Among these, the first two, *B* and *FLO*, are observed by photographic plates and only thin tracks, following the nomenclature of *B*, are counted. *FLO* discriminate between showers occuring in light and heavy nuclei and notice that the angular distribution is considerably different for each case. For heavy nuclei the estimated energy ~6 Bev is too low to reconcile with the minimum energy of primaries of about 6 Bev (see § 4.1 and 6), considering the rather flat energy spectrum in this region. On the other hand, the estimated energy ~12 Bev for light nuclei is not inconsistent with the average energy of primaries. From this fact, the assignment of the absolute value of the energy by the average angular spread leads, for showers produced in heavy materials, to a serious error.

Furthermore, there are many sources of error in cloud chamber and counter work. The average angle of *F* is of course too large. Beside the above source, the error comes from the mixing of fast protons in penetrating particles. We can correct this effect accounting for the distribution of grey tracks in *B* and obtain the average angle ~13°. This seems to give a reasonable value, but one must not forget the fact that a considerable number of particles with large angular spread escape observation and these particles contribute greatly to the average angle. In the counter work, *W*, the average angle is surprisingly small, which means that almost all protons may be absorbed by the lead filter between the two lowest counter trays and the large angle ones may escape. Then the average angle except for photographic works serves only to give the relative order of the energy, such as the comparison of *F* and *BM*.

In photographic works, however, we can discuss detailed properties. One remarkable point is the forward restriction of the secondary particles of light stars in *FLO*. If the energy of emitted mesons is unique in c.m.s., the angle is limited by the relation $\sin\theta < p_0^*/\xi\gamma\beta\mu$. Putting the maximum angle as 60° and $\xi \sim 2$, $p_0^*/\mu > 3/2$. If a meson is emitted backward with this momentum, this track should be observed as a grey one. This contradiction suggests that we cannot accept either the unique energy emission or the single collision in a nucleus or both. There is another piece of evidence against the single collision even in a light nucleus. Some of the light stars have an asymmetric angular distribution, which is suggested by *FLO* as a possible indication of a spin dependent interaction in a single collision. But some of these stars have two or more groups of secondary particles. This fact may not be due to the spin dependent interaction, but may be due to the plural collisions in a nucleus, although the experimental evidence is not so clear as to permit the decision of the above alternative.

The angular distribution in the single collision is observed in a star obtained by Freier and Ney.[24] The produced mesons, 10 or 11, are noticeably grouped around the two emitting angles, $\sim 1°$ and $\sim 10°$. These angles are transformed into $\sim 12°$ and $\sim 168°$ in c.m.s., if we adopt the energy of the incident nucleon as ~ 60 Bev as estimated by them on the basis of the relativistic increase of the ionization. This represents the narrow angle emission in forward and backward. However, this cannot be the evidence for the anisotropic emission, because there may be a considerable fluctuation and the method of estimation of the energy is not free from ambiguity

The anisotropic emission is also inferred from the analysis about the core of extensive air showers.[25] But this does not necessarily mean the anisotropic emission of mesons, if we consider that the scattered nucleon plays an important role in the development of the core. Then the scattered angle of the secondary nucleon must be small, as is the case of vector or pseudovector potential (see §3).

5. *Multiplicity*. It has been disputed whether or not the genuine multiple production of mesons exists. The recent experiments seem to have settled this problem in such a way that genuine multiple production certainly occurs at extremely high energies together with plural production.[5,6,7,24] Even in the moderate energy region, the multiple process is not necessarily neglected as discussed by several authors.[6,34,26]

At what energy multiple production becomes predominant is an interesting problem. We shall see this from the distribution of observed multiplicity ν. In the photographic works[6,17,27] and in counter works,[23,38] we can find the integral ν distribution as $\sim \nu^{-2}$ for $\nu < 10$, but in other experiments one sees a considerable deviation from ν^{-2}. For example, we see $\sim \nu^{-2.5}$ in *BM* and ν^{-3} in *F*. Such deviation is supposed to be due to the bias in the experimental conditions, if we notice the fact that in *W* and *P* the factor of about 2 is multiplied by the observed ν in order to get the true number of shower particles. In the cloud chamber experiments, the particles with small and large divergent angles may be missed, because of the low optical resolving power in the former case and of the limited illuminating region and the counter selection in the latter case. If we tentatively multiply the factors 1.5 and 2 by the observed ν in *BM* and *F*, respectively, accounting for the looser counter selection in *BM*, the corrected ν-distribution agrees with $\sim\nu^{-2}$. The agreement becomes better, if the small contribution from grey tracks is accounted.

ν^{-2} distribution is regarded as representing the approximate proportionality between ν and the primary energy, though the energy distribution is slightly less steep than E^{-2}. Then we can use ν as indication of energy as seen in § 4.1. This behavior, including the slight difference as mentioned above, is well explained by the plural production theory of Heitler and Janossy.[29] Then we may consider that the meson is singly produced as far as ν^{-2} distribution holds. The ν-distribution becomes steeper for $\nu \gtrsim 10$, which means the set-in of the genuine multiple production. Actually the multiple production may begin at smaller ν than 10 since both the theory of Heitler and Janossy and the experiments give a gradual bend in this region. Then we adopt $\sim$10 Bev as the critical energy where multiple production becomes predominant.

If we substitute this critical energy, corresponding to $W^*=2.5$ Bev, in (3.4) for $n=1$, we get the relation

$$(g\eta)^2/4\pi=3\pi(\mu/W^*)^2=3.0\times10^{-2}. \tag{4.1}$$

To verify our theory in § 3, we see the relation between n and $\bar{\theta}$, assuming $\bar{\theta}=1/\bar{\xi}$, in an individual star with many thin tracks. The most systematic data are given by *FLO*. We plot the ν-$(1/\bar{\theta}-1)$ relation in Fig. 1, together with the formula (3.6) and the result of Osborne's theory. The heavy and light stars are completely separated as emphasized by *FLO*, though they show the similar inclination. If we examine each star in detail, three light stars, which have the large values of $\nu^{3/2}/(\bar{\xi}-1)$, seem to have two or more cores, and one heavy star, which has small $\nu^{3/2}/(\bar{\xi}-1)$ show a single core, though other heavy stars may have two or more cores. If we adopt the lowest value of $\nu^{3/2}/(\bar{\xi}-1)$, a good estimation of $g\eta$ will be obtained:

$$(g/\eta)^2/4\pi=(\pi/4)n^{-3}(\mu/W^*)^2\approx1.0\sim2.2, \tag{4.2}$$

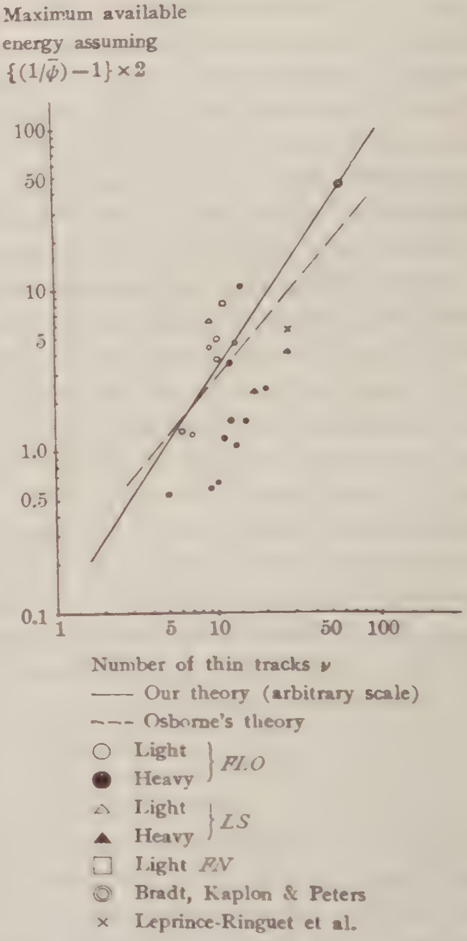
accounting for the contribution of neutral mesons as in (3.9). The estimated value $(g\eta)^2/4\pi$ is about 50 times larger than that in (4.1).

The reasons for such a discrepancy are considered as follows. (1) η is not a constant and increases with incident energy. (2) ν is still larger than N in the above estimation, so that the plural collision takes place almost always even in a light nucleus. (3) $\bar{\theta}$ is smaller than the above estimated value because of the same reason as in (2).

The magnitude of η can be estimated from the absorption of nucleons in the atmosphere. In order to explain the longer absorption mean free path than the collision mean free path, about a half of the energy of a nucleon must remain after a collision with an air nucleus.[16] This estimate holds for nucleon energies near 10 Bev, which form essentially the bulk of cosmic ray intensity. This amount of the surviving energy corresponds to $\eta\sim1/2$, provided the average number of collisions in an air nucleus is two. Therefore, the increase of η affects the result at most 4 times, even if η become unity in the higher energy region.

The other two causes are more sensitive, since both make the result larger in the same direction. If we divide the secondary particles into two groups, $\bar{n}$ decreases about one half and $\bar{\theta}$ decreases two or more times. Such a posibility is fairly plausible, if one looks at the distribution of shower particles in detail. For this reason, the estimate (4.2) is not reliable enough to obtain the magnitude of the coupling constant g. This presumption will be supported by considering the parallel behavior between heavy and light stars in Fig. 1. On the contrary, the estimate (4.1) has an ambiguity of at most a factor 2.

Fig. 1. Relation between multiplicity and angular divergence



Referring to (4.1) and assuming $\eta=1/2$, we obtain

$$g^2/4\pi\approx0.12. \tag{4.3}$$

This figure is not inconsistent with that obtained from the analysis of the experiment of artificial mesons.[30]

6. *Nuclear interaction of secondary particles.* To interpret the behavior of secondary particles, their constitution is different from each other according to the experimental conditions. The energy range, in which the protons and mesons are identified as fast shower particles, is shown in Table II for each observer.

Table II. The energy range of fast particles

Observer	Proton		Meson	
	Minimum energy Mev	Minimum range in Pb g cm^{-2}	Minimum energy Mev	Minimum range in Pb g cm^{-2}
*B**	330	124	49	18.5
BM	230	78	34	11.6
*F***	120	14.5	41	14.5

The constitution of both components is much different according to the energy range. Fowler showed that about 80% of the thin tracks with energies greater than the above values are mesons but that in the lower energy region this ratio is reversed.[31),17)] Accordingly a large part of the penetrating secondary particles in the cloud chamber photographs may be protons. In the counter work of Piccioni,[28)] almost all should be mesons following his description, though there is some objection to his interpretation.[32)]

The mean free paths for the nuclear interaction of secondary particles obtained by several authors are summarized in *B*. They criticized the results of other works and stated that the mean free path is the geometrical one. One should be careful to draw any conclusion from this result about the nuclear interaction of mesons, since the interaction varies with the energy of the meson.

Firstly, we consider the nuclear scattering of a meson by a free nucleon. We may ignore the damping effect, since we consider meson energies of at most a few Bev.[33)]

Accounting for the straightforward perturbation calculation, the pseudoscalar coupling gives the larger cross-section than the geometrical one if we adopted $f^2/4\pi=16$, while for pseudovector coupling the cross-section approaches the geometrical one at $\epsilon_0\sim M$ if we adopted $g^2/4\pi=0.1$. In the former case the splitting of a meson is possible even for the ordinary energy $\epsilon_0\sim\mu$ to M, while in the latter case the splitting may not take place and the nuclear interaction should rarely be observed.

Nevertheless, the actual situation is much different, because a meson strikes a compound nucleus. If the momentum of an incident meson is smaller than 200 Mev/*c*, corresponding to the maximum momentum of a nucleon in a nucleus, the simple elastic scattering should almost be forbidden by the Pauli exclusion principle for a recoil nucleon. This effect will considerably reduce the scattering cross-section of mesons with energies of several hundred Mev.

However, the cross-section for nuclear interaction will not be so greatly reduced, since there is a competing process, the absorption of a meson by a nucleus.* This process is just the inverse process of the production of a meson in a nucleon-nucleon collision.[34] This cross-section is larger than that for production for the same matrix element, because of the large density of final states, in which only two nucleons are present in the former case, whereas two nucleons and one meson are produced in the latter case. The absolute value of the cross-section is largely dependent on the magnitude of the coupling constant, but if we adopt the experimental value for the production cross-section, the absorption cross-section is the order of $\sim 10^{-27}$ cm^2. Here one must notice the fact that the magnitude of the coupling constant $f^2/4\pi$ determined from this experiment is much smaller than that obtained from the cross-section for meson production by γ-rays.[30] If we adopt the latter value of f, the cross-section becomes very large, about 10^{-25} cm^2. If we refer to the result of B, the absorption process is more probable than the others.

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* Other photographic works, *LS* and *FLO*, use about the same discrimination.

** Most of cloud chamber works except *BM* discriminate between the tracks by the penetration of a lead plate without electronic interaction.

§ 5. Conclusion

Although we cannot draw definite conclusions from the above analysis because of poor statistics in the experiments, we may say, referring to II, that the mesonic interaction is dependent upon the energy of the colliding nucleon as follows.

Below 2 Bev elastic scattering is more predominant than meson production. Above 2 Bev meson production is still single, though we observe multiple mesons in a nuclear collision which are attributed to the plural collisions of a nucleon in a nucleus. At about 10 Bev genuine multiple meson production sets in. The behavior of the multiple production can be explained by our theory I, but we cannot rule out other theories at the present stage of our knowledge.

The existence of multiple production strongly suggests the necessity of pseudovector coupling of a pseudoscalar meson, which is supported by various arguments. Pseudoscalar coupling is not always necessary and may not exist or may be weak if it does exist. One of the arguments for this assumption is the contradiction, which appears when we estimate the magnitude of the pseudoscalar coupling constant in the meson production by γ-rays or protons. If we take into account only pseudovector coupling in the case of proton bombardment, the magnitude of the pseudovector coupling constant is estimated as $g^2/4\pi \sim 0.2$. This value is consistent with the case of γ-ray bombardment and with our estimate in the present paper. However, this magnitude seems to be too large to evaluate the probability of various processes by the simple perturbation theory. If we retain our above estimate, we may fall into self-inconsistency, because our theory is grounded on the perturbation theory. The development of other approximation methods, for example, the improvement of the strong coupling theory, is strongly required.

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* This process was emphasized by Dr. Steinberger in a conversation with one of the authors (S. H.).

We should like to express our sincerest thanks to Professor Feld and Mr. Lebow, who gave us their experimental material and valuable advice.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On the Applicability of the Method of the Mixed Fields in the Theory of the Elementary Particles

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(Received August 3, 1950)

§ 1. Introduction and Summary.

The characteristic of the elementary particles, as was definitely shown by Yukawa in his theory of the meson, is their *mutual transformability*. Hence, in the world of the elementary particles, it is meaningless to ask what particles are simple and what particles are complex. For example, though a neutron transforms itself into a proton and a negative Yukawa particle, it is incorrect to say that the former is composed of the latter particles. As a proton transforms itself likewise into a neutron and a positive Yukawa particle, it may also be justified to state that the proton is a complex particle composed of a neutron and a positive Yukawa particle. Thus the concept of *simplicity* is complementary rather than contradictory to the concept of *complexity* in the world of the elementary particles. Cosequently, even in the case when we investigate the behaviour of each elementary particle separately, the existence of all the other elementary particles which interact with it must be taken into account at the same time. It was just for this reason that we emphasized in our earlier paper[1] the importance of *the method of the mixed fields* in the theory of the elementary particles.

The fundamental principle of the method of the mixed fields, which was there described was based upon the view that *part of difficulties which appeared in the current theory was due to the overlooking of intimate relations between the elementary particles.* In the history of the development of the theory of elementary particles, predictions of unknown particles were always made by using this method consciously or unconsciously. The prediction of the meson by Yukawa as early as in 1934 was just the case. Later on, one of the present author[2] also adopted consciously this method in order to overcome the difficulties of the meson theory, and predicted the existence of two sorts of mesons having different masses, before the discovery of π- and μ-mesons in cosmic rays.

Since 1945, we have studied the quantum electrodynamics from this standpoint in order to clarify how far the method of mixed fields can be used for solving the divergent difficulties which are inherent in the present formalism. First the self-energy of the electron due to the electromagnetic field was investigated(3).

In order to obtain a finite result, we had suggested the necessary existence of a cohesive force field described by a neutral scalar field, whose quanta were called *C*-mesons. In fact, if we set up a relation like

$$f^2=2e^2 \tag{1.1}$$

between the coupling constant f of the *C*-meson with the electron and the elementary charge e, the diverging term in the self-energy vanishes at least in e^2-approximation, and the remaining terms can be amalgamated into the empirical mass of the electron. A similar conclusion was also obtained independently by Pais(4) in his theory of f-field. Subsequently, a very promising aspect of this theory was revealed by Itoh, Koba and Tomonaga.(5) They proved that the introduction of the *C*-meson could overcome the serious difficulties appeared in the elastic scattering of the electron by a potential field whose important rôle in the quantum electrodynamics was first pointed out by Bethe and Oppenheimer(6)

Next, we attacked the problem of the vacuum polarization by taking into account the existence of various sorts of charged particles, in accordance with our methodological standpoint(7). In this way, it was found that the self-energy of the photon could be made zero, if we assume the following relations between the various sorts of charged particles:

$$\begin{gathered} N^{(0)}-2N^{(1/2)}+3N^{(1)}=0, \\ \sum_{i=1}^{N^{(0)}}(\mu_i^{(0)})^2-2\sum_{i=1}^{N^{(1/2)}}(\mu_i^{(1/2)})^2+3\sum_{i=1}^{N^{(1)}}(\mu_i^{(1)})^2=0 \end{gathered} \tag{1.2}$$

where $N^{(0)}$, $N^{(1/2)}$ and $N^{(1)}$ denote respectively the number of sorts of the charged scalar (including pseudoscalar), spinor and vector (including pseudovector) particles which exist in nature, and $\mu_i^{(0)}$, $\mu_i^{(1/2)}$ and $\mu_i^{(1)}$ respectively the mass of the i-th sort of the above mentioned types of particles. In spite of this success, the induced charge diverges as usual showing the limit of applicability of the method of mixed fields.

Meanwhile, considerable progress in the quantum electrodynamics was made by the application of Tomonaga-Scwinger's complete covariant formalism with the renormalization procedure. The divergent self-energy and induced charge of the electron, as had been ingeniously proved by them, could respectively be amalgamated into its observed mass and charge in a covariant manner. In spite of the great success of this theory in explaining the Lamb-Retherford shift of the hydrogen atom as well as the anomalous magnetic moment of the electron, it becomes immediately clear that there remain yet some more difficulties. Especially, the most important defect is the problem of the self-energy of the photon(8), which turns out to be indefinite owing to the occurrence of yet undefined mathematical symbol, as was pointed out by Pauli and Villars(9). Furthermore, it should be remarked that the renormalization procedure could be justified only if the quantities to be amalgamated were finite. Moreover, the defect of this formalism becomes more serious when applied to the meson theory.

In our oppinion, the Tomonaga-Schwinger formalism is considered in some way to be an *abstract* formalism, behind which the physical *model* of the elementary particles lies hidden. Here, the word "*model*" means not only the intimate relations between the elementary particles which can be revealed by the method of mixed fields, but also their internal structures. So long as the model merely plays the rôle of warranting the correctness of the procedures (e. g. renormalizations of mass and charge) involved in Tomonaga-Schwinger formalism, it is not necessary to know their detailed features. But, as soon as the defect of the formalism becomes obvious, it is required to obtain the more precise knowledge about the model.

Now, in our earlier paper[(1)], we stressed the importance of standing upon *the historical and realistic view point*, in order to find out the right direction of the theoretical progress. As we mentioned, the indispensability of this view point was first pointed out by Taketani[10] in his excellent works on the history of science. According to him, our cognition of nature is a spiral process repeating the circle corresponding to the three stages of advance, which are called respectively *phenomenological, substantialistic and essentialistic* stages. In the first stage, separate experimental facts are described by themselves. Next, comes the stage in which the structure of the object giving rise to these phenomena is clarified. Finally, we arrive at the highest stage, in which it is made clear what phenomenon a system with a given structure causes under given conditions. Looking upon the present state of the theory of elementary particles from such a view point, it seems to be in a state of searching for a path to sublimate itself from the substantialistic to the essentialistic stage, preliminarily clarifying the model of the elementary particles. We believe that the usefulness of the method of mixed fields in the present state certainly owes to these circumstances. Therefore, it should be remarked that *the method of mixed fields is based essentially on the historical and realistic view point.* Here, "*historical*" means that it is only effective in the beginning of the substantialistic stage, and "*realistic*" means that all the fields introduced have to be considered as really existing in nature.

In order to overcome the remaining difficulties in the Tomonaga-Schwinger formalism, an ingenious method of regulator has been proposed by Pauli and Villars[(9)] who stated this method as a *formalistic* generalization of the method of mixed fields. Though all the results thus obtained turn out to be finite and definite, there appears again a new kind of indefiniteness owing to the multiplicity of the regularization procedure. As far as the *formalistic* view point is taken, direct reference to the experiments is the only way to get rid of this indefiniteness. But, such a *phenomenological* aspect seems to us so unsatisfactory that we never consider it as a final solution. If we want to find out the physical background ensuring the uniqueness of the regularization procedure, the introduction of a *realistic* view point will again be required. However, as investigations by Umezawa et al.[(7)], Rayski et al.[(11)], and Feldman[(12)] show, no success has yet been achieved by the mixture of the *real* fields in removing at once all the difficulties from the present formalism. For example, the induced charge produced by pair formation turns out by no means to be finite, unless we admit the introduction of imaginary coupling constants.

In spite of these circumstances, we don't think that the formalistic view point is preferable to the realistic one, because, from the *historical* and *realistic* standpoint which was mentioned above, it is not surprising that the method of the mixed fields which can reveal the *external* and *accidental* connections between the elementary particles is merely applicable in the beginning of the *substantialistic* stage of theoretical advance, and fails in solving those problems which lie beyond the limit of its applicability. In order to solve the latter problems, it will be required to elevate the theory into a higher stage where the *internal* and *inevitable* connections between the elementary particles, or, in other words, *the structures of the elementary particles*, are found. The theory of the *non-local* fields which has recently been proposed by Yukawa(13) seems to be very promising as an attempt tending this direction.

But, whatever the final solution may be, it seems worthwhile at the moment to determine the limit of the applicability of the method of mixed fields, in order to clarify where truly the problems to be solved lie.

§ 2. The Theory of the *C*-Meson.

As is well known, the most fundamental difficulties appeared in the quantum theory of the wave fields are the diverging self-energy of the elementary particles and the problem of the vacuum polarization. Though a remarkable progress was recently made by the advent of Tomonaga-Schwinger's complete covariant formalism with renormalization procedure, the fundamental difficulties were *avoided* rather than *solved* by this formalism. In fact, as was mentioned in § 1, the renormalization procedure could only be justified, if these difficulties were truely solved. As far as this formalism gives finite results in conformity with the experiments, it may be regarded as an *abstract formalism*, behind which a *concrete model* lies hidden. But, as the defect of the formalism becomes obvious, we must look for the precise structure of the model.

As a possible model which makes the self-energy of the electron finite, we proposed the theory of the *C*-meson. This theory was then extended to all the other charged particles (proton, π-meson, μ-meson, and so on), though the relations between the coupling constants obtained in the Bose-Einstein cases seem to be somewhat artificial(3)(4).

In the first place, we shall discuss the consequences of the theory of the *C*-meson which can be compared with the experiments. At first sight, it seemed that the theory of *C*-meson would give a reasonable explanation for the mass difference between the proton and the neutron, and hence we had expected that the mass of the *C*-meson could be determined from this fact(3)(4). But, this attempt proved afterward to be by no means successful, because the best value, which could be obtained if the C-meson would have a vanishing mass, amounted only to 75% of the observed mass difference(14). Nevertheless, this result is rather favourable, because the mass of the C-meson previously determined in this way was so small that it was irreconcilable with the other facts. As Wightman(15) pointed out, the energy difference of the mirror nuclei can no longer be understood if the mass of the C-meson is smaller than $276m$, where m denotes the mass of the electron. Also, Zilsel, Blair and Powell(16) found from the analysis of the proton-proton scattering that the C-meson with a mass between $40m$ and $140m$ considerably affects the symmetrical property of the nuclear forces.

If the C-meson has a mass far greater than that of the electron, its contributions to the Lamb-Retherford shift as well as the anomalous magnetic moment of the electron are negligible as compared with the radiative corrections in conformity with the experiments(17)(18). The C-mesonic correction to the elastic scattering of the electron, as was shown by Koba et al.(19), is also negligible. Generally speaking, as long as the Compton wave length of the C-meson is assumed to be far smaller than that of the electron, the existence of C-meson can not give rise to any detectable effect in the electronic phenomena.

The effect of the C-meson will probably, become conspicuous in those phenomena in which the proton, π-meson, or μ-meson takes part. But, in the cases of the proton and π-meson, the C-mesonic effects lie often hidden behind those of the nuclear fields. In fact, the anomalous magnetic moment of the proton due to the interaction with the C-meson turns out to be far smaller than that which is due to the nuclear interactions, though it may be comparable to the radiative correction.

We may expect the creation of the C-mesons in the high energy collisions. But, as the C-meson is considered to decay into an electron pair with a rest lifetime of the order of 10^{-21} sec, the creation of the C-meson will always be observed as that of the electron pair. First, the creation of a C-meson by the collision of a high energy photon with a charged spinor particle was calculated(20), but the result turned out to be very small as compared with that of the ordinary pair creation. Next, the creation by the Brems-C-strahlung of a charged particle was discussed(20)(21). This effect becomes considerable only when the mass of the incident particle is comparable to or larger than that of the C-meson. But, as the absorption curve of the cosmic rays underground which consist mainly of the μ-mesons, has been explained correctly by taking into account electromagnetic effect alone, the Brems-C-strahlung of the μ-meson must be negligibly small. From this fact, it is concluded that the mass of C-meson is larger than $500m$(21). We may expect that the C-meson with such a large mass will be found in the mixed shower of the cosmic rays or in the high energy nuclear phenomena.

Next, we shall investigate the *self-stress* of the electron in the light of the theory of the C-meson(22). As Pais stressed in his excellent book(23), it should be required in a consistent relativistic theory of the electron that not only the self-energy is finite but also the self-stress is *zero*. The latter was required so that the self-energy may have a transformation property of the inertia mass. The self-energy of moving electron, which was calculated directly by using the Tomonaga-Schwinger formalism or the usual perturbation theory with relativistically improved integration method (w-method)(24), has just the required transformation property, nevertheless the self-stress of the electron at rest is finite. As Pais and Epstein(25) pointed out, this conflict is closely related to the divergent difficulty of the self-energy.

Thus we might expect that the difficulty of the non-vanishing self-stress would be solved by the introduction of the C-meson too. But, actually, it was shown by Yukawa and Umezawa(22) that this expectation was partly true. According to them, we can derive the following relation between the self-stress S and the self-energy μ of the electron at rest:

$$S=-\frac{1}{3}\left(\mu-m\frac{\partial\mu}{\partial m}\right)+S_{V.P.}, \tag{2.1}$$

where m denotes the mass of the electron. This relation is the generalization of that obtained by Pais and Epstein. The first term in the right hand side of (2.1) represents the contribution from the processes which give rise to the self-energy, whereas the second term, denoted as $S_{V.P.}$, is that which is due to the process in which pairs of the charged particles take part as in the case of the vacuum polarization. In the following we shall call them respectively the self-stress of the *mass type* and that of the *vacuum polarization type*. In the usual quantum electrodynamics, where the electron interacts only with the electromagnetic field, the self-stress of vacuum polarization type vanishes identically, justifying Pais-Epstein's formula. But, this term cannot be ignored for the C-meson.

In the theory of the C-meson, we can easily show that self-stress of the mass type, which was a stumbling-block for the usual quantum electrodynamics, becomes zero, as was expected(22). But, instead of it, the diverging self-stress of vacuum polarization type appears due to the interaction with C-meson. Nevertheless, it should be remarked that, though the difficulty was not removed, its nature was entirely changed. The difficulty in the theory of the C-meson is quite similar to that of the vacuum polarization which we shall discuss in the next section.

Now, we shall discuss the self-energy of the C-meson due to the virtually created pairs of the charged particles(7). Though, it is the exact counterpart of the self-energy of the photon, which we shall also discuss in the next section, this difficulty cannot be removed by taking into account the existence of the various charged particles at the same time. However, we are not yet in a position to state anything definite about the self-energy of the C-meson, because its rôle in the theory of the nucleons has never been clarified. In this connection, it is interesting to remark that this divergence can easily be removed, if we assume the existence of two kinds of C-meson having the same nature and interacting directly with each other.

Finally, we shall add a remark concerning the higher order approximation of the perturbation theory. The self-energy of the electron in the fourth order approximation was calculated by the method of Feynman and Dyson, but it could never been made finite by the introduction of the C-meson alone(26). Nevertheless, this result is by no means unexpected, because it is a matter of course that the difficulties of new types connected with the vacuum polarization will appear successively, as we proceed to the higher order approximations. Therefore, the more we raise the order of approximation, the more kinds of elementary particles must be taken into account simultaneously.

§ 3. The Problem of the Vacuum Polarization.

In the quantum theory of the wave fields, the problem of the vacuum polarization is confronted with two types of difficulties, one of which is the appearance of non-gauge-invariant terms in the observable quantities (non-vanishing self-energy of the photon, etc.), and the other is the occurrence of divergence (infinite self-charge, etc.). In this section, we shall review this problem from our methodological standpoint stated in § 1, and show how far these difficulties could be removed by the application of the method of mixed fields.

According to the well known calculation in the Tomonaga-Schwinger theory, the current induced in vacuum by an external electromagnetic field described by the potential A_μ^{ext} (whose source we shall denote $J_\mu^{ext}=-\frac{1}{4\pi}\Box A_\mu^{ext}$) is given by(9)

$$\delta J_\mu=-4e^2\int K_{\mu\nu}(x-x')A_\nu^{ext}(x')d\omega' \tag{3.1}$$

where $K_{\mu\nu}$ is the world tensor of the second rank which consists bilinearly of the invariant delta functions (and its derivatives) belonging to the charged particle virtually created in vacuum, and the integration is extended over the entire four-dimensional volume x', whose volume element we denote $d\omega'$. For the gauge-invariancy of this expression, it is required that the condition $\frac{\partial K_{\mu\nu}}{\partial x_\nu}=0$ is satisfied. But, actually, this condition is by no means valid, and instead we obtain(9) (11) (12)

$$\frac{\partial K_{\mu\nu}}{\partial x_\nu}=n\,\delta(x)\frac{\partial \varDelta^{(1)}(x)}{\partial x_\mu} \tag{3.2}$$

with

$$n=\begin{cases} +1 & \text{(for charged scalar particle),} \\ -2 & \text{(for charged spinor particle),} \\ +3 & \text{(for charged vector particle).} \end{cases} \tag{3.3}$$

As Pauli(9) emphasized, the left hand side of (3.2) is indeterminate due to the singularity of $\frac{\partial \Delta^{(1)}}{\partial x_\mu}$ on the light cone which has the form $\frac{x_\mu}{(x_\lambda x_\lambda)}$. The occurrence of such an undefined mathematical symbol seems to be a stumbling-block in the development of the Tomonaga-Schwinger theory.

When the external field is that of the photon, the appearance of the non-gauge-invariant term in δJ_μ leads at once to the difficulty of non-vanishing photon self-energy in conflict with experiments. Moreover, the calculated value of this self-energy, as was shown by many authors(7) (8) (9) (11), depends on the method of integration. The cause of this ambiguity may also be considered to originate from the singular character of the invariant delta function.

In order to get rid of these difficulties, we shall introduce the realistic standpoint that all the charged particles existing in nature should be considered at the same time. Then, $K_{\mu\nu}$ becomes the sum of those belonging to each of the really existing charged particles, and we obtain instead of (3.2)

$$\frac{\partial K_{\mu\nu}}{\partial x_\mu}=\delta(x)\frac{\partial}{\partial x_\mu}\left\{\sum_{i=1}^{N^{(0)}}\Delta^{(1)}(x,\mu_i^{(0)})-2\sum_{i=1}^{N^{(1/2)}}\Delta^{(1)}(x,\mu_i^{(1/2)})+3\sum_{i=1}^{N^{(1)}}\Delta^{(1)}(x,\mu_i^{(1)})\right\} \tag{3.2'}$$

where $N^{(0)}$, $N^{(1/2)}$ and $N^{(1)}$, and $\mu_i^{(0)}$, $\mu_i^{(1/2)}$ and $\mu_i^{(1)}$ have the same meaning as those defined in § 1, and $\Delta^{(1)}(x,\mu)$ denotes the $\Delta^{(1)}$-function belonging to the particle with mass μ. From (3.2′) we can easily see that the difficulty as regard to the non-gauge-invariance is removed, at least in e^2-approximation, by assuming relations (1.2) between the charged particles existing in nature(7) (11) (12).

In this connection, we shall add some remarks about the relation to the correspondence theoretical treatment of the vacuum polarization in the theory of the positron developed by Dirac(27) and Heisenberg(28). According to Weisskopf(29), the physical significance of this theory is summarized into the three postulates, stating respectively that the following quantities are meaningless:

(1) The energy of the vacuum electron in the field-free space,
(2) The charge and current densities of the vacuum electron,
(3) A temporally and spatially constant, field-independent polarizability of the vacuum.

Here, the postulates (1) and (2) are the definition of the vacuum, while the postulate (3) means the renormalization of the charge.

In order to make the definition of the vacuum gauge-invariant, it was remarked by Weisskopf(29) that one must take as the energy of the vacuum electron which should be subtracted

$$U_0=-2/(2\pi)^3\int\left\{\sqrt{(\vec{p}-e\vec{A})+m^2}-eA_0\right\}d\vec{P} \tag{3.4}$$

instead of the usual zero point energy of the electron, where $\vec{P}$ denotes the momentum of the vacuum electron and the integration is extended to all the momentum space. Further A_0 and $\vec{A}$ denote respectively the scalar and vector potential of the external electromagnetic field. Now, if we expand (3.4) in the power of e, the e^2-term turns out to be identical with the self-energy of the photon obtained by the usual perturbation theory. From this relation, we may see that the problem of gauge-invariance is closely connected with the consistent definition of the vacuum(30).

The vacuum energy for any sort of charged particle is generally given by(30)

$$U=n/(2\pi)^3\int\{\sqrt{(\vec{P}-e\vec{A})^2+\mu^2}-eA_0\}\,d\vec{P} \tag{3.4'}$$

where n is defined by (3.3) and μ denotes the mass of the charged particle concerned. From this expression, it is immediately seen that, if we take into account all the charged particles existing in nature simultaneously, and assume (1.2) as well as some additional relations being valid for them, the total vacuum energy vanishes identically. Hence, in such a case, it becomes unnecessary to subtract the vacuum energy.

Next, we shall discuss the properties of the induced current (3.1) more precisely. By carring out the integration, we may write (3.1) as the following form

$$\delta J_\mu=(a_2+a_1\Box+a_3\Box^2+\cdots\cdots)J_\mu^{ext} \tag{3.5}$$

where the non-gauge-invariant term discussed above is omitted, and a_1, a_2, a_3, ⋯ ⋯denote numerical constants. The first term in the right hand side of (3.5), which changes the strength of the original current by a factor $1+a_1$, should be considered as being already included in the observed external current. That is, we must renormalize the charge of the incident particle by taking into account this factor. This procedure corresponds exactly to the postulate (3) of the correspondence theoretical treatment mentioned above.

The actual evaluations of a_i for the charged particles of scalar, spinor and vector types were performed by many authors(7) (11) (12) (31), but the results have given rise to serious difficulties. Firstly, it was shown that the charge renormalization factor a_1 diverges for each case, and moreover depends on the nature of the charged particle concerned. Though Tomonaga-Schwinger theory avoided these difficulties by enforcing the charge renormalization, this procedure will be valid only when they were truly solved.

For the solution of divergencies mentioned above, the method of mixed fields is of no use, because the renormalization factor a_1 has minus sign in all the cases, including higher spin particles and, moreover, though the divergences in the scalar and spinor cases are logarithmic, the quadratic divergence appears in the vector case(7)(12). The introduction of the tensor coupling for the spinor as well as for the vector cases cannot improve this bad situation(12). The cancellation between the different orders of approximations seems also hopeless(32). We believe that this difficulty lies surely beyond the limit of the applicability of the method of mixed fields.

On the contrary, the realistic stand point is very favourable for the solution of the latter difficulties mentioned above, because, in this case, the renormalization factor always becomes the total sum of those for each of the really existing charged particles without regard to the incident particle(7)(32).

Finally, it should be remarked that a_2 diverges in the spinor case with tensor coupling as well as the vector case with vector coupling, and a_2 and a_3 diverge in the vector case with tensor coupling(7)(12)(31). These difficulties can neither be *avoided* by Tomonaga-Schwinger theory nor be *solved* by the method of the mixed fields.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

A New Interpretation of the Negative Energy Bose Field

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(Received August 5, 1950)

§ 1. Introduction

The well-known difficulties inherent in the negative energy Bose fields limit the unambiguous application of such fields in the ordinary quantum field theory. A correct incorporation of the negative energy Bose fields into the ordinary theory of the positive energy Bose fields, however, seems to give a promising method in order to solve the convergence difficulties of the quantum field theory. For examples, in Bopp's field(1) or Pauli's regulator method,(2) it seems to be an essential point of the theory that the negative energy fields or the fields having the imaginary constants of coupling with the real fields are mixed with the positive energy fields. As stated by Pauli, the current mixed theory of ordinary fields may stand on the realistic point of view, on the other hand his regulator method may stand on the formalistic point of view.

Although the regulator method could be considered to be a generalized mixed field theory as mentioned above, it will be necessary that such fictitious fields have to be eliminated by some limiting procedures in the final results which make them to be only formal auxiliary fields and to have no correspondece to any observable one.

The introduction of a new hypothesis into the fictitious fields, however, will allow us to infer that it would be possible to bring about the reconciliation of the realistic point of view and the formalistic one, and the particles being accompanied by the field would appear to be as real an existence as any other particles known in physics. Recently, the present author tried to introduce a hypothesis into the theory of negative energy Bose fields(3).

A brief consideration was presented, but it is noy yet to give a full account on it. Now, we shall give detailed accounts of the underlying idea, which is a close analogon to Dirac's positron theory.

§ 2. An assembly of similar systems

Any quantized field, e.g. Yukawa's field, is equivalent to a system of identical Bose particles, i.e. Yukawa's particles.

As well-known, a state of the field will be determined by giving the distribution of all numbers n_i of the particles which occupy the i-th proper state. In the ordinary field theory, it is sufficient to consider only one system of the identical particles in order to consider the behaviour of that field.

The problem we must now consider is how to extend the field concept for the situation of the negative energy Bose particles.

In place of considering only one system of the identical particles which represents the quantized field, let us consider an assembly of the similar systems of the particles which represent one state of the field respectively. We assume that a statistics is satisfied by this assembly whose element is not a particle, but the whole system of the particles having the given statistical properties in accordance with that given quantized field.

An arbitrary statistics of the assembly could be assumed without any contradiction, e.g. Bose or Fermi statistics.

If only one system of assembly should exist in nature everywhere, the statistics of the assembly would be superfluous and yield no more results than that obtainable in the ordinary theory.

The assumption of the assembly of positive energy Bose field, therefore, seems to be of a superfluous redundancy. The introduction of the assembly and the statistics, however, will be expected to give an essential change of the situation for the negative energy Bose fields.

Up to present, several models of negative energy Bose fields are presented. But we here consider the most simple one as follows.

At first, let us consider a scalar field U. It satisfies the well-known field equation:

$$(\Box-\varkappa^2)\,U=0, \tag{1}$$

and it has the positive definite Hamiltonian density H.

Then, let us introduce a scalar field V which satisfies the same equation as the field U:

$$(\Box-\varkappa^2)\,V=0 \tag{2}$$

and has such a negative definite Hamiltonian density $-H$ that U are interchanged by V in the Hamiltonian density H of U and the negative sign -1 is multiplied.

The quantized energy of the V field, therefore, will be given by

$$\left.\begin{aligned} -\bar{H}=-\sum_i n_i E_i, \quad & E_i=\sqrt{k_i^2+\varkappa^2} \\ & i=1,\,2,\ldots\ldots \\ & n=0,\,1,\,2,\ldots\ldots \end{aligned}\right\}, \tag{3}$$

where k_i denotes the i-th proper wave vector as usual.

Consequently, the particles accompanied by the V field may be considered as the negative energy Bose particles which seem to have no correspondence to nature.

If the particles would obey to the Fermi statistics, the ingenious method of Dirac's positron theory would save the difficulties. For the Bose particles, however, Dirac's method is not applicable.

Here, we introduce the assembly of the negative energy Bose field as mentioned above and assume that the assembly satisfies the Fermi statistics. Let us denote the variables of the j-th system of the assembly by $n_i^{(j)}$. We consider the configuration space of m systems:

$$\begin{array}{l} (n_1^{(1)},\ n_2^{(1)},\ n_3^{(1)},\dots\dots\dots\dots) \\ (n_1^{(2)},\ n_2^{(2)},\ n_3^{(2)},\dots\dots\dots\dots) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (n_1^{(m)},\ n_2^{(m)},\ n_3^{(m)},\dots\dots\dots\dots). \end{array}$$

Then we can apply Fock's second quantization method for this assembly in the configuration space of $n_i^{(j)}$. This gives us the quantized wave functions $\phi(n_1, n_2, \dots\dots; t)$ and $\phi^+(n_1, n_2, \dots\dots; t)$ which satisfy the following commutation relations

$$[\phi^+(n_1, n_2, \dots\dots; t),\quad \phi(n_1', n_2', \dots\dots; t)]_+ = \delta_{nn'}, \tag{4a}$$

where $\delta_{nn'} = \delta_{n_1 n_1'}\,\delta_{n_2 n_2'},$

$$[\phi(n_1, n_2, \dots\dots; t),\quad \phi(n_1', n_2', \dots\dots; t)]_+ = 0\ , \tag{4b}$$

and

$$[\phi^+(n_1, n_2, \dots\dots; t),\quad \phi^+(n_1', n_2', \dots\dots; t)]_+ = 0. \tag{4c}$$

According to Pauli's exclusion principle, this assembly gives such distributions of systems that two systems could not occupy at the same time the same state of the system, i.e., the identical distribution of particles can not occur in two systems.

§3. The definition of a vacuum

For the negative energy Bose field and also the positive energy Bose field, we assume that the fields should be given by the Fermi assembly of the system of quantized field particles. A perfect vacuum will be a region where the following conditions are satisfied.

I. For a positive energy Bose field, only the zero particle state, in which all n_i's are equal to zero, is occupied.

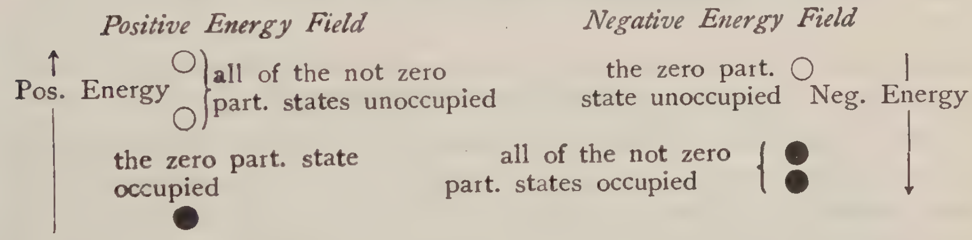
II. For a negative energy Bose field, the zero particle state is unoccupied and all other states are occupied.

This assumption requires there to be a distribution of one positive energy system and infinite negative energy systems everywhere in the world. For the positive energy system, there is no contradiction with the ordinary theory as stated before. On the other hand, the negative energy systems have a very analogy to the negative energy sea of a vacuum of Dirac's particles. The interaction of the negative energy system with an other field may give a transition of one of the occupied states of a vacuum to the unoccupied zero particle state. For example, the system which holds single negative energy particle, e.g. $n_i=1$ and other $n_{j\neq i}=0$, may pass into the zero particle state; all $n_i=0$. Such a transition may appear as a creation of single positive energy particle.

Thus the unoccupied states of a vacuum are possible to be interpreted as the system of real observable partcles.

A vacuum state may be schematically elucidated as Fig. 1.

Fig. 1



Positive Energy Field

Negative Energy Field

Pos. Energy
all of the not zero part. states unoccupied
the zero part. state occupied
the zero part. state unoccupied
Neg. Energy
all of the not zero part. states occupied

Some detailed applications of our theory to the current problems of the field theory will appear in the later issues of this journal.

We wish to express our cordial thanks to Prof. H. Yukawa for his guidance and encouragement through out our study life, and also to Prof. S. Sakata for his kind interest in this work.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Note on a Lorentz-Invariant Integration in the Quantum Field Theory*

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(Received August 7, 1950)

Resumo. Unue ni esploras la kondiĉon de integrebleco en integrata formo de super-multa-tempa teorio, kaj due proponas novan esprimaĵon por transformada funkcio, kiu estas komplete Lorentz-nevarianta. Plue ni aplikas ĉi tiun formon je la kalukulado de S-matrikso kaj klarigas la ĝeneralan egalecon de Tomonaga-Schwinger'a formalaĵo al tiu de Feynman. Fine unu provo de integra formuligo de kvantuma kampteorio estas diskutata por ebligi estontan ĝeneraligon de la teorio.

§ 1. Introduction and summary.

After the first stage of rapid developments of his meson theory, Yukawa[1] gave from a most general standpoint a critical review regarding the basis of the current theory of elementary particles. One of the main points which he emphasized in this article was that the quantum field theory of that time was not yet formulated in a completely Lorentz-covariant manner, and in this respect he suggested that the concept of the general transformation function once proposed by Dirac[2] would play an important rôle in the future expansion of the theory.

In response to Yukawa's proposition, Tomonaga[3] soon published his relativistically covariant formulation of the quantum theory of wave-fields, the so-called super-multiple-time theory, and showed that the transformation function $U(\sigma, \sigma_0)$, which transforms a state vector $\Psi(\sigma_0)$ into $\Psi(\sigma)$, possesses those properties as Dirac's general transformation function should have.

Some years later Schwinger[4] independently completed the same formalism and Dyson[5] could express the transformation function U, including the S-matrix as a special case, of the quantum electrodynamics in a most compact and practical form. He owes his success to the introduction of certain concepts and calculation rules of Feynman's[6] theory, which had grown out of a quite different idea, namely a space-time approach to quantum field theory, into the Tomonaga-Schwinger theory.

In this note we shall try to give a simple covariant expression of the transformation function or S-matrix, which is valid not only in the quantum electrodynamics but also in various meson theories, where Dyson's formula would suffer from the complication due to the normal-dependent part of the interaction Hamiltonian density. Our expression resembles in its form the general transformation function of Dirac most closely and, as we hope, confirms on one hand Yukawa's suggestion to a certain amount and on the other hand completes the proof of the equivalence between the Tomonaga-Schwinger and the Feynman theories, which has been carried out in quantum electrodynamics by Dyson[5].

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* The main content of this work was read before the Symposium on the Theory of Elementary Particles held on December 9, 1949, in Tokyo.

We are aware that our method here described is still insufficient in many respects and must be further scrutinized before it can rest on a firm basis. Nevertheless it might be allowed for us to make here a rough sketch of it, deferring more fundamental considerations till a later opportunity, because we hope that our method represents at least a convenient empirical rule for the calculation of S-matrix, which is of practical importance.

In the first place we examine the well-known integrability condition of the field theory in its integral form (§2); this will clarify the physical meaning of that condition and the rôle of the normal dependent part of the Hamilton density function. Then we give a formula to construct an interaction Hamilton density, which satisfies the integrability condition, out of a given Lagrange function (§3). On the basis of these considerations we then propose an invariant expression for the transformation function, making use of a modified P-symbol and the Lagrange density function, prove its equivalence to the usual expression (§4). Further an application of this formula to the S-matrix is studied and simple practical rules for its calculation described (§5). Finally the current quantum field theory is summarized in an integral form and the possibility of generalizing it to a non-local theory is discussed (§6).

2. Derivation of the integrability condition.

When the super-multiple-time theory was founded[2], it was made clear that an integrability condition must hold for the interaction Hamiltonian density, and further, when this formalism was applied to meson theories[7][8], the general form of that condition was found to be*

$$\left[i\hbar\frac{\delta}{\delta\sigma(x)}-H(x,\boldsymbol{n}(x)),\quad i\hbar\frac{\delta}{\delta\sigma(x')}-H(x',\boldsymbol{n}(x'))\right]=0, \qquad (2\cdot1)$$

where x and x' are any two points on a space like surface σ, $\boldsymbol{n}(x)$ and $\boldsymbol{n}(x')$ denoting the unit vectors normal to the surface at x and x' respectively. (In the coordinate system which is tangential to σ at x, the components of the vector $\boldsymbol{n}(x)$ turn out $n_k=0$ $(k=1,2,3)$, $n^0=-n_0=1$.)

In the following we derive the same condition alternatively from the uniqueness of the integrated solution. Although the calculation then becomes more complicated, the physical meaning of the requirement of integrability can be seen more clearly, and it leads naturally to the covariant integration method to be described in the next section.

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* We use the unit $c=1$ throughout this note.

It may be noticed that the integrability condition is to be imposed on the Hamilton density function also in the case of Schrödinger or Heisenberg representation[9] and even in the classical theory of wave field, as long as the latter is expressed in the covariant canonical formalism. In this note, however, we restrict ourselves only to the interaction representation.

Given a state vector $\Psi(\sigma_I)$ on a certain "initial" space-like surface σ_I and the Hamilton density function $H(x, n(x))$, one can at once write down the state vector $\Psi(\sigma_F)$ on any "final" surface σ_F. For that purpose one may introduce, according to Dyson[5], a one-parameter family of space-like surfaces $\{\sigma_i\}$ between σ_I and σ_F, in such a way that there is one and only one member through every point. These surfaces act the rôle of the time, and the relation such as "prior", "posterior" or "simultaneous", between any two points is completely fixed. Consequently one can define the well-known chronological operator P without ambiguity.

The solution of the Tomonaga-Schwinger equation is then expressed in the form

$$\Psi(\sigma_F)=U(\sigma_F, \sigma_I)\Psi(\sigma_I), \tag{2·2}$$

$$U(\sigma_F, \sigma_I)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-i}{\hbar}\right)^m \int_{\sigma_I}^{\sigma_F} d^4x_1 \int_{\sigma_I}^{\sigma_F} d^4x_2 \dots \int_{\sigma_I}^{\sigma_F} d^4x_m$$

$$\times P[H(x_1, n(x_1)),\ H(x_2, n(x_2)), \cdots\cdots\cdots, H(x_m, n(x_m))], \tag{2·3}$$

the convergence of the series being assumed. On the right hand side the integrands depend, in general, on the surface, but since we have fixed the set of space-like surfaces, these integrals have definite meaning.

The expression (2·3) depends, therefore, implicitly on the intermediate surfaces $\{\sigma_i\}$. But we could as well derive $\Psi(\sigma_F)$ from $\Psi(\sigma_I)$ by making use of another set of surfaces $\{\sigma_i'\}$ between σ_I and σ_F, and the result should be identical with that obtained through $\{\sigma_i\}$; otherwise the $\Psi(\sigma_F)$ could not be determined uniquely and the causal relations would be lost. Thus we require that the right-hand side of (2·3) should be invariant, except a trivial phase factor, under any space-like deformation of the intermediate surfaces. This is the condition which a Lorentz-invariant theory must satisfy without fail, because the latter property is exhibited not in the employment of curved surfaces or a generalized time, but in the fact that the prediction of the theory is independent of the choice of the reference system.

It is necessary and sufficient to find out the condition that when a certain intermediate surface σ_j undergoes an infinitesimal deformation at a point x_j, the transformation function should remain essentially unchanged. We have, for the general term of (2·3),

$$\frac{\delta}{\delta\sigma_j(x_j)}\left\{\frac{1}{m!}\left(\frac{-i}{\hbar}\right)^m\int_{\sigma_I}^{\sigma_F}d^4x_1\int_{\sigma_I}^{\sigma_F}d^4x_2\dots\int_{\sigma_I}^{\sigma_F}d^4x_m\, P(H(x_1,n(x_1)),\right.$$
$$\left.\times H(x_2,n(x_2)),\dots,H(x_m,n(x_m)))\right\}$$

$$=\frac{1}{(m-1)!}\left(\frac{-i}{\hbar}\right)^m\int_{\sigma_j}d^3x'_j\int_{\sigma_I}^{\sigma_F}d^4x_1\dots\int_{\sigma_I}^{\sigma_F}d^4x_{m-1}\, P\Big(\frac{\delta H(x'_j,n(x'_j))}{\delta\sigma_j(x_j)},$$
$$H(x_1,n(x_1)),\dots,H(x_{m-1},n(x_{m-1}))\Big)$$
$$+\frac{1}{2}\frac{1}{(m-2)!}\left(\frac{-i}{\hbar}\right)^m\int_{\sigma_j}d^3x'_j\int_{\sigma_I}^{\sigma_F}d^4x_1\dots\int_{\sigma_I}^{\sigma_F}d^4x_{m-2}\, P\,([H(x'_j,n(x'_j)),$$
$$H(x_j,n(x_j))],H(x_1,n(x_1)),\dots,H(x_{m-2},n(x_{m-2}))),\qquad(2\cdot4)$$

where x'_j denotes a variable point on the particular surface σ_j. The first term of (2·4) is due to the change of the integrand through its normal-dependent terms, while the second comes from the rearrangement of the operators according to the definition of the P-symbol, since a point near x_j, which lay before deformation on the future side of x'_j, is now found on the past side. (Fig. 1). Summing up (2·4) with respect to m, one obtains

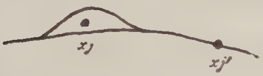

Fig. 1.

$$\frac{\delta U(\sigma_F,\sigma_I)}{\delta\sigma_j(x_j)}$$
$$=-\frac{i}{\hbar}\int_{\sigma_j}d^3x'_j\sum_{m=0}^{\infty}\left(\frac{-i}{\hbar}\right)^m\int_{\sigma_I}^{\sigma_F}d^4x_1\dots\int_{\sigma_I}^{\sigma_F}d^4x_m$$
$$P(Q(x_j,x'_j),H(x_1,n(x_1)),\dots,H(x_m,n(x_m))),\qquad(2\cdot5)$$

with

$$Q(x_j,x'_j)=\frac{\delta H(x'_j,n(x'_j))}{\delta\sigma_j(x_j)}-\frac{i}{\hbar}\frac{1}{2}[H(x'_j,n(x'_j)),H(x_j,n(x_j))].\qquad(2\cdot6)$$

If, therefore, $Q(x_j,x'_j)$ is a c-number function, we have

$$\frac{\delta U(\sigma_F,\sigma_I)}{\delta\sigma_j(x_j)}=-\frac{i}{\hbar}c(x_j,\sigma_j)\,U(\sigma_F,\sigma_I)\qquad(2\cdot7)$$

where $c(x_j,\sigma_j)$ is a c-number commuting with all operators, and U is altered by a trivial phase factor. This conclusion can be at once extended to finite deformation and U then remains essentially independent of the choice of $\{\sigma_i\}$.

We shall put for simplicity $c=c(\sigma)$, and this condition is summarized, written in a symmetrical form, as

$$Q(x_j,x'_j)-Q(x'_j,x_j)=0.\qquad(2\cdot8)$$

or $$\frac{\delta H(x'_j, n(x'_j))}{\delta\sigma_j(x_j)} - \frac{\delta H(x_j, n(x'_j))}{\delta\sigma_j(x'_j)} - \frac{i}{\hbar}[H(x'_j, n(x'_j)), H(x_j, n(x_j))]$$
$$= \frac{i}{\hbar}\left[i\hbar\frac{\delta}{\delta\sigma_j(x_j)} - H(x_j, n(x_j)), i\hbar\frac{\delta}{\delta\sigma_j(x'_j)} - H(x'_j, n(x'_j))\right] = 0, \quad (2\cdot9)$$

which is identical with (2·1).

From the derivation above described, it is easily understood why the interaction Hamiltonian density of the non-covariant one-time theory just satisfies the integrability condition, only if it is expressed in a covariant manner. For we know that the evaluation of the S-matrix (interaction energy is assumed to vanish at $t=\pm\infty$) in the one-time theory gives a unique result independent of the intermediate reference system (for example i) or ii) in Fig. 2).

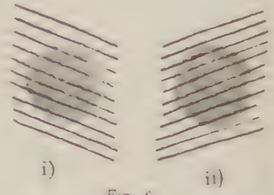
i) ii)

Fig. 2.

§3. Construction of the interaction Hamilton density function.

It is to be remarked that the integrability condition is in principle always satisfied, as long as the canonical formalism is consistent, because the uniqueness of $U(\sigma_F, \sigma_I)$ except a phase factor, can be proved from indirect considerations. For example one can set up a system of Lorentz-covariant equations of motion for the wave fields in the Heisenberg picture, not referring to canonical variables, and if one specifies the wave fields (with their derivatives if necessary) on the initial space-like surface σ_I, one can completely determine the wave fields at every point of the space-time, including of course those on σ_F. But this means that in the Schrödinger or interaction representation the state vector $\Psi(\sigma_F)$ is uniquely determined from $\Psi(\sigma_I)$, the only arbitrariness lying in the phase factor. (A similar line of reasoning has been followed by de Wet[10], making use of a covariant canonical formalism*.)

But when we wish to extend the theory to cases where the usual canonical formalism fails, the integrability condition will appear as a guiding principle, or at least as a criterion, for the determination of the interaction Hamilton density function. In these cases a systematic rule to construct such a function will become useful.

It was pointed out previously by Kanesawa and the author[11] that the interaction Hamilton density function, which satisfies the integrability condition, can be constructed from the interaction term of the Lagrange function by supplementing it successively with suitable normal-dependent terms. This statement has been verified in several cases investigated so far.

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* The author wishes to thank Mr. G. Takeda, who has informed me of de Wet's article and has made it available for him.

Here we shall give a simple formula to determine these additional terms.

$$H(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x})) = \sum_{m=0}^{\infty} H_m(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x})), \tag{3·1}$$

with

$$H_0(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x})) = -L(\boldsymbol{x}), \tag{3·2}$$

....

$$H_m(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x})) = -\frac{1}{2^m}\left(\frac{-i}{\hbar}\right)^m \int^{\sigma(x)} d^4\boldsymbol{x}_1 \int^{\sigma(x_1)} d^4\boldsymbol{x}_2 \cdots\cdots \int^{\sigma(x_{m-1})} d^4\boldsymbol{x}_m$$

$$\times [L(\boldsymbol{x}), L(\boldsymbol{x}_1), \ldots\ldots, L(\boldsymbol{x}_m)]. \tag{3·3}$$

In the above formula the square bracket is the abbreviation of repeated commutators according to the notation of Hausdorff⁽¹²⁾,

$$\left.\begin{aligned} &[A] \equiv A, \\ &[A, B] \equiv AB - BA, \\ &[A, B, C] \equiv [[A, B], C], \\ &[A, B, C, D] \equiv [[[A, B], C], D], \\ &\cdots\cdots \quad \cdots\cdots \\ &[A, B, C, \ldots\ldots, Z] \equiv [[\ldots[[A, B], C], \ldots], Z]. \end{aligned}\right\} \tag{3·4}$$

The integrations on the right hand side of (3·3) are extended only over the space-like regions with regard to the point $\boldsymbol{x}$. $\sigma(\boldsymbol{x})$ is the space-like surface through $\boldsymbol{x}$; $\sigma(\boldsymbol{x}_1)$ denotes a space-like surface passing both $\boldsymbol{x}$ and $\boldsymbol{x}_1$ and lying on the past side of $\sigma(\boldsymbol{x})$; $\sigma(\boldsymbol{x}_2)$ means a space-like surface through $\boldsymbol{x}$ and $\boldsymbol{x}_2$ and lying posterior to $\sigma(\boldsymbol{x})$ and $\sigma(\boldsymbol{x}_1)$; and so on. (Fig. 3). Since one has only to do with the infinitesimal neighbourhood of $\boldsymbol{x}$, further characterization of the surfaces $\sigma(\boldsymbol{x}_1)$, $\sigma(\boldsymbol{x}_2)$,...... is not necessary.

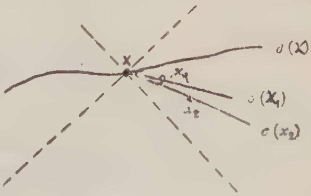

Fig. 3.

In the expression (3·3) for $H_m(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}))$ we are concerned with m points, any two of which are found at an infinitesimal space-like distance, and so we are forced to specify their mutual positions more clearly. Let us take this occasion of introducing a set of notations to express the relative positions of space-time points. We shall agree:

$$\left.\begin{aligned} &\boldsymbol{x}_1 > \boldsymbol{x}_2 \text{ means "}\boldsymbol{x}_1\text{ lies in a future time-like position to }\boldsymbol{x}_2\text{".} \\ &\boldsymbol{x}_1 < \boldsymbol{x}_2 \text{ means "}\boldsymbol{x}_1\text{ line in a past time-like position to }\boldsymbol{x}_2\text{".} \\ &\boldsymbol{x}_1 \sim \boldsymbol{x}_2 \text{ means "}\boldsymbol{x}_1\text{ lies at a finite space-like distance from }\boldsymbol{x}_2\text{".} \\ &\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \text{ means "}\boldsymbol{x}_1\text{ lies at an infinitesimal space-like distance} \\ &\qquad\text{from }\boldsymbol{x}_2\text{".} \end{aligned}\right\} \tag{3·5}$$

The generalization to several points is trivial except for the case of those lying infinitesimally close together. We introduce, in analogy to (3·4), the symbol

$$(x \circ x_1 \circ x_2 \circ \cdots\cdots \circ x_m) \equiv ((\cdots((x \circ x_1) \circ x_2) \circ \cdots) \circ x_m), \tag{3·6}$$

to represent the situation which is obtained by the following procedure: one starts from m points lying at finite space-like distances to one another, first lets x_1 approach x up to an infinitesimal distance, then lets x_2 draw up to the doublet $(x \circ x_1)$, then lets x_2 come near to the triplet $((x \circ x_1) \circ x_2)$, and so on, at last to realize an assemblage of m points situated in an infinitesimal region. A different situation for m points is of course possible, but we need only the above mentioned case.

By the help of this notation, (3·3) can be expressed more rigorously as

$$H_m(x, \boldsymbol{n}(x)) = -\frac{1}{2^m}\left(\frac{-i}{\hbar}\right)^m \int_{(x \circ x_1)}^{\sigma(x)} d^4 x_1 \int_{(x \circ x_1 \circ x_2)}^{\sigma(x_1)} d^4 x_2 \cdots \int_{(x \circ x_1 \circ \cdots \circ x_m)}^{\sigma(x_{m-1})} d^4 x_m$$
$$\times [L(x), L(x_1), \ldots\ldots, L(x_m)]. \tag{3·3}$$

There remains still ambiguity concerning the lower limit of the integration. This we eliminate by the help of the adiabatic prescription introduced by Tati and Tomonaga:[13)]

$$\int^{\sigma} F(x) d^4 x = \int_{\sigma} G_\mu(x) d\sigma^\mu, \tag{3·7}$$

where G is defined by

$$G_\mu(x) = \frac{1}{(2\pi)^2} \int F(k) \frac{k_\mu}{(k_\lambda k_\lambda)} e^{i(kx)} d^4 k, \tag{3·8}$$

with

$$F(k) = \frac{1}{(2\pi)^2} \int F(x)\, e^{-i(kx)} d^4 x. \tag{3·9}$$

It amounts to putting the contribution of the lower limit equal to zero.

In this way the Hamilton density function $H(x, \boldsymbol{n}(x))$ is perfectly defined. It is not difficult to verify that this function really satisfies the integrability condition (2·1). But perhaps we had better give its elementary but tedious proof at the end of this note. (Appendix I).

§ 4. Lorentz-invariant integration of the Tomonaga-Schwinger equation.

The investigation of the integrability condition in § 2 has shown that the characteristic rôle of the normal-dependent terms in $H(x, \boldsymbol{n}(x))$ consists in the successive cancellation of the acausal effects that arise from the non-commutativity of $H(x, \boldsymbol{n}(x))$'s at two space-like points. Thus the essential contribution to the integration seems to come from the invariant terms of the interaction Hamiltonian, the other terms are used only to eliminate the ambiguity concerning the intermediate surfaces. It appears, therefore, that the addition of the normal dependent terms, which is often a tedious procedure, and the final reelimination of their effects, which has been so often experienced in meson theories, is a rather cumbersome detour.

On the other hand the rule described in § 3—though half-empirical as yet—for the construction of the interaction Hamilton density function from the interaction Lagrangian suggests that the above mentioned essential part would be in general the interaction Lagrange function itself.

Extrapolating these considerations, we would assume that an invariant expression for the transformation function using only the Lagrange function would be possible, if the erasure of the acausal effects could be guaranteed by a suitable device. In the following we shall give an attempt to this purpose.

First we notice that Dyson's P-symbol is not, in general, a completely Lorentz-invariant conception, namely when two of its argument operators refer to two space-like points and are yet non-commutative. We therefore introduce a modified symbol

$$P^*(A(x_a), B(x_b), C(x_c), \ldots\ldots, Z(x_z)),$$

which is defined, by the help of the notation (3·5) and (3·6), as follows.*

i) When any two of the points $x_a, x_b, \ldots\ldots, x_z$ are situated at time-like positions to each other, P^*-symbol is identical with P-symbol. For example, when

$$x_a > x_b > \ldots\ldots > x_z,$$

$$P^*(A(x_a), B(x_b), \ldots, Z(x_z)) \equiv A(x_a)B(x_b)\ldots Z(x_z). \qquad (4\cdot1)$$

ii) When some of the points $x_a, x_b, \ldots, x_z$ are situated at space-like positions to each other with finite distances, P^*-symbol is defined symmetrically with respect to these points, but as the corresponding operators commute, the result is effectively identical with that of a P-symbol. For instance, when

$$\begin{aligned} & x_a > x_b > \ldots > x_h > (x_i \sim x_j) > x_k > \ldots > x_z, \\ & P^*(A(x_a), b(x_b), \ldots, H(x_h), I(x_i), J(x_j), K(x_k), \ldots Z(x_z)) \\ \equiv & \frac{1}{2}\{AB\ldots HIJK\ldots Z + AB\ldots HJIK\ldots Z\} \\ = & AB\ldots HIJK\ldots Z = AB\ldots HJIK\ldots Z; \end{aligned} \qquad (4\cdot2)$$

and when

---

* The P^*-symbol for two argument operators was given previously by us[14]. Also a somewhat different definition has been employed by Nishijima[15]. We thank Mr. Nishijima for having sent us a copy of his work before publication.

$$x_a > \{(x_b > x_c) \sim x_d\} > x_e > \dots > x_z,$$

$$P^*(A(x_a), B(x_b), C(x_c), D(x_d), E(x_e), \cdots Z(x_z))$$

$$\equiv \frac{1}{3}\{ABCDE\dots Z + ABDCE\dots Z + ADBCE\dots Z\}$$

$$= ABCDE\dots Z = ABDCE\dots Z = ADBCE\dots Z. \qquad (4\cdot 3)$$

iii) When some of the points $x_a, x_b, \dots x_z$ are situated at space-like positions to each other with infinitesimal distances, P^*-symbol is defined symmetrically with respect to these points, and as the corresponding operators do not commute in general, the result is different from that of a P-symbol. For instance, when

$$x_a > x_b > \dots > x_h > (x_i \circ x_j) > x_k > \dots > x_z,$$

$$P^*(A(x_b), B(x_b), \dots, H(x_h), I(x_i), J(x_j), K(x_k), \cdots Z(x_z))$$

$$\equiv \frac{1}{2}\{AB\dots HIJK\dots Z + AB\dots HJIK\dots Z\}$$

$$\neq AB\dots HIJK\dots Z; \qquad (4\cdot 4)$$

and when

$$x_a > x_b > \dots > x_h > (x_i \circ x_j \circ x_k) > x_e > \dots > x_z,$$

$$P^*(A(x_a), B(x_a), \dots, H(x_h), I(x_i), J(x_j), K(x_k), L(x_l), \dots, Z(x_z))$$

$$\equiv \frac{1}{4}\{AB\dots HIJKL\dots Z + AB\dots HJIKL\dots Z$$

$$+ AB\dots HKIJL\dots Z + AB\dots HKJIL\dots Z\}; \qquad (4\cdot 5)$$

and when

$$(x_a \circ x_b \circ \dots \circ x_m) > x_n > \dots > x_z,$$

$$P^*(A(x_a), B(x_b), \dots, M(x_m), N(x_n), \dots, Z(x_z))$$

$$\equiv \frac{1}{2^{m-1}}\{A, B, \dots, M\}\, N\dots Z, \qquad (4\cdot 6)$$

where the curly bracket is used, in analogy to (3·4), to represent repeated anticommutators:

$$\{A, B\} \equiv AB + BA,$$

$$\{A, B, C, \dots, M\} \equiv \{\{\dots\{\{A, B\}, C\}, \dots\}, M\}. \qquad (4\cdot 7)$$

From the foregoing it is obvious that the P^*-symbol is defined quite independently of any set of space-like surfaces, so that it is a completely Lorentz-invariant notion.

Making use of this P^*-symbol we propose to express the solution of the

Tomonaga-Schwinger equation in the following form*,

$$U(\sigma_F, \sigma_I) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i}{\hbar}\right)^m \int_{\sigma_I}^{\sigma_F} d^4 x_1 \int_{\sigma_I}^{\sigma_F} d^4 x_2 \ldots \int_{\sigma_I}^{\sigma_F} d^4 x_m$$

$$\times P^*(L(x_1), L(x_2), \ldots, L(x_m)), \tag{4·8}$$

where $L(x)$ is the interaction part of the Lagrange density function and is of course a world scalar. Thus (4·8) is perfectly independent of the intermediate surfaces, since P^* as well as $L(x)$'s have nothing to do with them. Consequently the uniqueness of the transformation function is evident and one need not take any notice of an integrability condition.

To prove that (4·8) indeed coincides with the usual solution, we show, i) U satisfies the boundary condition

$$U(\sigma_I, \sigma_I) = 1, \tag{4·9}$$

and ii) the infinitesimal variation of U is determined by the Tomonaga-Schwinger equation**. The first point can be verified at once. We shall then set up from (4·8) the differential equation for $U(\sigma, \sigma_I)$

$$i\hbar \frac{\delta U(\sigma, \sigma_I)}{\delta\sigma(x)} = -\sum_{m=1}^{\infty} \frac{1}{(m-1)!} \left(\frac{i}{\hbar}\right)^{m-1} \int_{\sigma_I}^{\sigma_F} d^4 x_1 \ldots \int_{\sigma_I}^{\sigma_F} d^4 x_{m-1}$$

$$\times P^*(L(x), L(x_1), \ldots, L(x_{m-1}))$$

$$= -\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i}{\hbar}\right)^m \int_{\sigma_I}^{\sigma_F} d^4 x_1 \ldots \int_{\sigma_I}^{\sigma_F} d^4 x_m \, P^*(L(x), L(x_1), \ldots, L(x_m))$$

$$= -\sum_{m=0}^{\infty} \left(\frac{i}{\hbar}\right)^m \sum_{r=0}^{m} \frac{(-1)^r}{2^r} \frac{1}{(m-r)!} \int_{(x \circ x_1)}^{\sigma(x)} d^4 x_1 \int_{(x \circ x_1 \circ x_2)}^{\sigma(x_1)} d^4 x_2 \ldots \int_{(x \circ x_1 \circ \ldots \circ x_r)}^{\sigma(x_{r-1})} d^4 x_r$$

$$\times [L(x), L(x_1), L(x_2), \ldots, L(x_r)] \int_{\sigma_I}^{\sigma_F} d^4 x_{r+1} \ldots \int_{\sigma_I}^{\sigma_F} d^4 x_m \, P^*(L(x_{r+1}), \ldots, L(x_m)) \tag{4·10}$$

The last equality can be understood as follows:

i) When any of the points $x_1, \ldots, x_m$ does not lie in the space-like neighbourhood of x, (see Fig. 4.),

$$P^*(L(x), L(x_1), \ldots, L(x_m))$$

$$= L(x) \, P(L(x_1), L(x_2), \ldots, L(x_m)), \tag{4·11}$$

---

* Yukawa[16] has pointed out that our expression appears in some respects related to the S-matrix in his non-local field theory.[17]

** The main feature of our proof is similar to that given by Nishijima,[15] to whom we owe much.

whence the first term $r=0$.

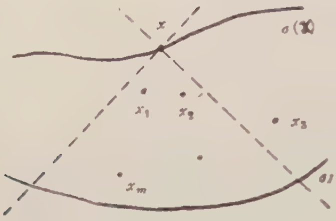

Fig. 4.

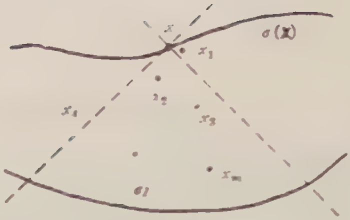

Fig. 5.

ii) When one of the m variable points, say x_1, lies space-like and infinitesimally close to x, (fig. 5.), then by definition

$$P^*(L(x), L(x_1), L(x_2), \ldots, L(x_m))$$

$$=\frac{1}{2} L(x)L(x_1)P^*(L(x_2), \ldots, L(x_m))$$

$$+\frac{1}{2} L(x_1)L(x)P^*(L(x_2), \ldots, L(x_m)). \qquad (4\cdot12)$$

Therefore in this region, one has to correct the expression $r=0$ (4·11) by the second term $r=1$. These terms together just make up the expression (4·12):

$$L(x)\, P^*(L(x_1), L(x_2), \ldots, L(x_m)) - \frac{1}{2}\, [L(x), L(x_1)]\, P^*(L(x_2), \ldots, L(x_m))$$

$$=\frac{1}{2}\, \{L(x), L(x_1)\}\, P^*(L(x_2), \ldots, L(x_m)). \qquad (4\cdot13)$$

iii) When further a second point, say x_2, approaches the doublet $(x \circ x_1)$ up to an infinitesimal distance, (fig. 6.), we have

$$P^*(L(x), L(x_1), L(x_2), L(x_3), \ldots, L(x_m))$$

$$=\frac{1}{4}\{L(x), L(x_1), L(x_2)\}P^*(L(x_3), \ldots, L(x_m)). \qquad (4\cdot14)$$

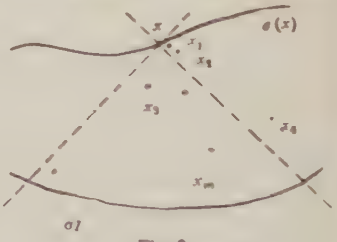

Fig. 6.

But in this region the first three terms $r=0, 1, 2$ yield just the same result:

$$L(x)\, P^*(L(x_1), L(x_2), L(x_3), \ldots, L(x_m)) - \frac{1}{2}\, [L(x), L(x_1)]$$

$$\times P^*(L(x_2), L(x_3), \ldots, L(x_m)) - \frac{1}{2}\, [L(x), L(x_2)]\, P^*(L(x_1), L(x_3), \ldots,$$

$$\times L(x_m)) + \frac{1}{4}\, [L(x), L(x_1), L(x_2)]\, P^*(L(x_3), \ldots, L(x_3))$$

$$=\Big\{\frac{1}{2}L(x)\{L(x_1),L(x_2)\}-\frac{1}{2}[L(x),L(x_1)]L(x_2)-\frac{1}{2}\ L(x),L(x_2)]L(x_1)$$
$$+\frac{1}{4}[L(x),L(x_1),L(x_2)]\Big\}P^*(L(x_3),\ldots,L(x_m))$$
$$=\frac{1}{4}\{L(x),L(x_1),L(x_2)\}P^*(L(x_3),\ldots,L(x_m)).\qquad(4\cdot15)$$

Similar relations can be derived for any $r\leqq m$.

Consequently we arrive at the conclusion,

$$i\hbar\frac{\delta U(\sigma,\sigma_I)}{\delta\sigma(x)}=-\Big\{\sum_{m=0}^{\infty}\frac{1}{2^m}\Big(\frac{-i}{\hbar}\Big)^m\int_{(x\circ x_1)}^{\sigma(x)}d^4x_1\ldots\int_{(x\circ x_1\circ\ldots\circ x_m)}^{\sigma(x_{m-1})}d^4x_m[L(x),Lx_1),\ldots,L(x_m)]\Big\}$$
$$\times\Big\{\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{i}{\hbar}\Big)^m\int_{\sigma_I}^{\sigma_F}d^4x_1\ldots\int_{\sigma_I}^{\sigma_F}d^4x_mP^*(L(x_1),\ldots,L(x_m))\Big\}$$
$$=H(x,n(x))\,U(\sigma,\sigma_I),\qquad(4\cdot16)$$

taking into account $(3\cdot1)-(3\cdot3')$. But this is just the Tomonaga-Schwinger equation for the transformation function.

§ 5. Evaluation of the S-matrix.

We shall now proceed to the evaluation of our invariant expression $(4\cdot8)$ for the case of S-matrix,

$$S\equiv U(\infty,-\infty)=\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{i}{\hbar}\Big)^m\int_{-\infty}^{\infty}d^4x_1\ldots\int_{-\infty}^{\infty}d^4x_mP^*(L(x_1),\ldots,L(x_m)),\quad(5\cdot1)$$

and establish some simple rules of calculation. Following Dyson's argument,[4] the estimation of an element of S-matrix referring to given initial and final states is reduced to evaluation of the vacuum expectation values of the expressions such as*

$$P^*(\phi(x_1),\phi(x_2)),\quad P^*\Big(\frac{\partial\phi(x_1)}{\partial x_{1\mu}},\phi(x_2)\Big),$$
$$P^*\Big(\frac{\partial\phi(x_1)}{\partial x_{1\mu}},\frac{\partial\phi(x_2)}{\partial x_{2\nu}}\Big).\qquad(5\cdot2)$$

Now the P^*-symbol containing two argument operators is explicitly expressed by[14]

$$P^*(A(x_a),B(x_b))\equiv\frac{1+\varepsilon^*(x_a,x_b)}{2}A(x_a)B(x_b)$$
$$+\frac{1-\varepsilon^*(x_a,x_b)}{2}B(x_b)A(x_a),\qquad(5\cdot3)$$

---

* As an example we take the case of neutral scalar field $\phi(x)$.

where the ε*-symbol means

$$\varepsilon^*(x_a, x_b) = \begin{cases} +1 & \text{for} \quad x_a > x_b, \\ 0 & \text{for} \quad x_a \sim x_b \quad \text{or} \quad x_a \circ x_b, \\ -1 & \text{for} \quad x_a < x_b, \end{cases} \tag{5·4}$$

which is again a completely Lorentz-invariant definition. It is clear that if one replaces in (5·3) the ε*-symbol by Schwinger's ε-symbol, one obtains the definition of *P*-symbol. Consequently, the vacuum expectation value of $P(\phi(x_1), \phi(x_2))$, say, is given essentially by

$$\Delta_F(x_1 - x_2) = \Delta^{(1)}(x_1 - x_2) + i\varepsilon(x_1, x_2)\Delta(x_1 - x_2), \tag{5·5}$$

while that of $P^*(\phi(x_1), \phi(x_2))$ is given by a modified function

$$\Delta_F^*(x_1 - x_2) = \Delta^{(1)}(x_1 - x_2) + i\varepsilon^*(x_1, x_2)\Delta(x_1 - x_2). \tag{5·6}$$

The two functions $\Delta_F(x_1 - x_2)$ and $\Delta_F(x_1 - x_2)$ are almost identical, since $\varepsilon^*(x_1, x_2)$ differs from $\varepsilon(x_1, x_2)$ only when $x_1 - x_2$ is a space-like vector, while $\Delta(x_1 - x_2)$ vanishes for such an $x_1 - x_2$. An appreciable distinction arises, however, when one differentiates these functions twice. In general the value of these singular functions at the origin contains ambiguity, but we can for the present employ the limiting value when the argument vector, always remaining space-like, approaches zero. Then we have

$$\frac{\partial \varepsilon(x, o)}{\partial x_0} = 2\delta(x_0), \tag{5·7}$$

but

$$\frac{\partial \varepsilon^*(x, o)}{\partial x_0} = 0\,; \tag{5·8}$$

so that

$$\frac{\partial \varepsilon(x, o)}{\partial x_0} \frac{\partial \Delta(x, o)}{\partial x_0} = -2\delta^4(x) \tag{5·9}$$

but

$$\frac{\partial \varepsilon^*(x, o)}{\partial x_0} \frac{\partial \Delta(x, o)}{\partial x_0} = 0. \tag{5·10}$$

Consequently one finds that

$$\frac{\partial \Delta^{(1)}(x_1 - x_2)}{\partial x_{1\mu}} + i\varepsilon^*(x_1, x_2)\frac{\partial \Delta(x_1 - x_2)}{\partial x_{1\mu}} = \frac{\partial \Delta_F^*(x_1 - x_2)}{\partial x_{1\mu}},$$

$$\frac{\partial^2 \Delta^{(1)}(x_1 - x_2)}{\partial x_{1\mu}\,\partial x_{1\nu}} + i\varepsilon^*(x_1, x_2)\frac{\partial^2 \Delta(x_1 - x_2)}{\partial x_{1\mu}\,\partial x_{1\nu}} = \frac{\partial^2 \Delta_F^*(x_1 - x_2)}{\partial x_{1\mu}\,\partial x_{1\nu}},$$

and, in general,

$$\frac{\partial^m \Delta^{(1)}(x_1 - x_2)}{\partial x_{1\alpha} \cdots \partial x_{1\mu}} + i\varepsilon^*(x_1, x_2)\frac{\partial^m \Delta(x_1 - x_2)}{\partial x_{1\alpha} \cdots \partial x_{1\mu}} = \frac{\partial^m \Delta_F^*(x_1 - x_2)}{\partial x_{1\alpha} \cdots \partial x_{1\mu}}. \tag{5·11}$$

This fact makes the expression of the vacuum expectation value quite simple, for one obtains

$$<P^*(\phi(x_1),\phi(x_2))>_{\rm vac}=\frac{\hbar}{2}\Delta_F^*(x_1-x_2), \tag{5·12}$$

as well as

$$\left\langle P^*\left(\frac{\partial\phi(x_1)}{\partial x_{1\mu}},\ \phi(x_2)\right)\right\rangle_{\rm vac}=\frac{\hbar}{2}\,\frac{\partial\Delta_F^*(x_1-x_2)}{\partial x_{1\mu}}, \tag{5·13}$$

$$\left\langle P^*\left(\frac{\partial\phi(x_1)}{\partial x_{1\mu}},\ \frac{\partial\phi(x_2)}{\partial x_{2\nu}}\right)\right\rangle_{\rm vac}=\frac{\hbar}{2}\,\frac{\partial^2\Delta_F^*(x_1-x_2)}{\partial x_{1\mu}\,\partial x_{2\nu}}; \tag{5·14}$$

and similar relations for higher derivatives, if such are needed. These circumstances are in marked contrast to the case of P-symbols, where one has, for example,

$$\begin{aligned}\left\langle P\left(\frac{\partial\phi(x_1)}{\partial x_{1\mu}},\ \frac{\partial\phi(x_2)}{\partial x_{2\nu}}\right)\right\rangle_{\rm vac}&=\frac{\hbar}{2}\left\{\frac{\partial^2\Delta^{(1)}(x_1-x_2)}{\partial x_{1\mu}\,\partial x_{2\nu}}+i\varepsilon(x_1,x_2)\frac{\partial^2\Delta(x_1-x_2)}{\partial x_{1\mu}\,\partial x_{2\nu}}\right\}\\&=\frac{\hbar}{2}\,\frac{\partial^2\Delta_F(x_1-x_2)}{\partial x_{1\mu}\,\partial x_{2\nu}}-2\,i\hbar\, n^\mu(x_1)\,n^\nu(x_2)\,\delta^4(x_1-x_2),\end{aligned} \tag{5·15}$$

taking account of (5·9). The second term in this expression is just compensated by the effect of normal-dependent terms of the interaction Hamiltonian, as was verified by Matthews15) through a straightforward calculation based on Dyson's formula. In our expression appears neither a δ^4-function nor a normal-dependent term, and so one can reach the result without being troubled by the presence of those terms, which are ultimately to be cancelled.

Thus we have obtained simple rules for the computation: i) one may employ the interaction Lagrangian $L(x)$, and no surface-dependent terms, ii) one may transfer the differentiation of the field quantity to Δ_F^*-function without any care, when evaluating vacuum expectation values.

The above argument shows at the same time that the results of the Tomonaga-Schwinger theory are perfectly identical with those of Feynman's formalism, where a normal-dependent term is out of question from the start.* Thus we hope to have completed the proof of equivalence between two theories for general cases, including various meson theories.

§ 6. Discussions.

Our expression (4·8) can be written symbolically

$$U(\sigma_F,\sigma_I)=\exp\frac{i}{\hbar}\int_{\sigma_I}^{\sigma_F}L(x)\,d^4x, \tag{6·1}$$

---

* Cf. also the recent work of Yang and Feldman.19)

where the exponential function containing non-commutative operators are defined by P^*-symbol:

$$\exp \frac{i}{\hbar} \int_{\sigma I}^{\sigma F} L(x)\, d^4x = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i}{\hbar}\right)^m \int_{\sigma I}^{\sigma F} d^4x_1 \ldots \int_{\sigma I}^{\sigma F} d^4x_m$$
$$\times P^*(L(x_1), \ldots, L(x_m)). \tag{6·2}$$

The right-hand side of (6·1) closely resembles in its form the general transformation function of Dirac, though the physical content is somewhat different. Also the differential equation (4·9) is symbolically expressed as

$$i\hbar \frac{\delta}{\delta\sigma(x)} \left\{\exp \frac{i}{\hbar} \int_{\sigma I}^{\sigma F} L(x')\, d^4x'\right\} = H(x, n(x)) \left\{\exp \frac{i}{\hbar} \int_{\sigma I}^{\sigma F} L(x')\, d^4x'\right\}, \tag{6·3}$$

which is similar to the well-known relation in the non-relativistic quantum mechanics,

$$i\hbar \frac{d}{dt} \left\{\exp \frac{i}{\hbar} \int_{\Delta t} L\, dt\right\} = H(t) \left\{\exp \frac{i}{\hbar} \int_{\Delta t} L\, dt\right\}, \tag{9·4}$$

where $\int_{\Delta t} L\, dt$ is the quantum analogue of the classical action function for small interval Δt.

These formal affinities would be surely not accidental, and we would say that the Lagrange function could be utilized in the quantum field theory in a more direct way than heretofore. Whether it is possible to construct a consistent scheme of the quantum field theory based on the Lagrange function, and to derive the above results from more fundamental relations, is now being studied.

Setting aside this basic problem, we shall assume for the present that (6·1) represents a relation resting on a secure foundation. Then we are led to summarize the main content of the current field theory in the following three postulates:

I) A system of interacting m wave fields, is described by a Lagrange density function:

$$L_{\text{tot}} = L_{(1)} + L_{(2)} + \ldots + L_{(m)} + L, \tag{6·5}$$

where $L_{(1)}, \ldots, L_{(m)}$ are Lagrange functions, which the first, second, ... m-th field would have when interaction tends to zero, and L represents the interaction of these fields.

II) The field quantities appearing in L obey the same equation of motion and the 4-dimensional commutation relations as when they are free.

III) In remote past and future L vanishes (or can be neglected) and one can define state vectors $\Psi(-\infty)$ and $\Psi(+\infty)$. (Nothing is assumed as to the intermediate $\Psi(\sigma)$'s.) These are connected by

$$\left.\begin{aligned} &\Psi(+\infty)=S\Psi(-\infty), \\ \text{with} \quad & \\ &S=\exp\frac{i}{\hbar}\int_{-\infty}^{\infty}L(x)\,d^4x. \end{aligned}\right\} \qquad (6\cdot6)$$

The exponential function is defined by (6·2).

This integral formulation evidently includes what the current theory, which requires the existence of intermediate $\Psi(\sigma)$'s, can predict as to transition processes. But if one starts from these integral postulates, it is comparatively natural for one to proceed to a generalized theory, which no longer assumes the presence of intermediate state vectors. Thus the path to the theory of non-local fields or of elementary particles with finite size, seems less steep in this S-matrix formalism by means of the Lagrange function. One trivial example of such an attempt is given in the appendix. (Appendix II).

This formalism is as yet tentative and leaves many points to be discussed. One of them, for example, is concerned with the 4-divergence term in the Lagrange function. As is well known, such a term has no effect at all as long as the Lagrange function appears only in the variation principle. If we utilize it, however, in a direct way as above mentioned, the existence of such a term gives rise to a certain change, which corresponds to a contact transformation in the usual theory. This would indicate that one should prefer some standard form of the Lagrangian to others. But any definite statement is certainly premature at present.

Acknowledgement. We wish to express our sincere thanks to Prof. K. Husimi, Dr. R. Utiyama, Prof. Y. Nambu, Mr. K. Nishijima and many others for their kind interest and valuable advices for this work.

Appendix I.

To prove that the constructed Hamiltonian

$$H(x,n(x))=\sum_{m=0}^{\infty}H_m(x,n(x)),$$

$$H_m(x,n(x))=-\frac{1}{2^m}\left(\frac{-i}{\hbar}\right)^m\int_{(x\circ x_1)}^{\sigma(x)}d^4x_1\ldots\int_{(x\circ x_1\circ\ldots\circ x)}^{\sigma(x_{m-1})}d^4x_m\;L(x),L(x_1),\ldots,L(x_m)], \qquad (3\cdot3')$$

Satisfies the integrability condition

$$i\hbar\frac{\delta H(x,n(x))}{\delta\sigma(x')}-i\hbar\frac{\delta H(x',n(x'))}{\delta\sigma(x)}+[H(x,n(x)),H(x',n(x'))]=0 \qquad (2\cdot1)$$

(x and x' being two points on σ).

From (3·1) and (3·3′) we have

$$[H(x, n(x)), H(x', n(x'))] = \sum_{m=0}^{\infty} \frac{1}{2^m} \left(\frac{-i}{\hbar} \right)^m K_m, \tag{I·1}$$

with

$$K_m = \sum_{r=0}^{m} \int_{(x \circ x_1)}^{\sigma(x)} d^4 x_1 \cdots \int_{(x \circ x_1 \circ \cdots \circ x_r)}^{\sigma(x_{r-1})} d^4 x_r \int_{(x' \circ x_{r+1})}^{\sigma(x)} d^4 x_{r+1} \cdots \int_{(x' \circ x_{r+1} \circ \cdots \circ x_m)}^{\sigma(x_{m-1})} d^4 x_m$$

$$\times [L(x), L(x_1), \ldots, L(x_r), [L(x'), L(x_{r+1}), \ldots, L(x_m)]]. \tag{I·2}$$

What we have to show is that

$$K_m = \int_{(x \circ x_1)}^{\sigma(x)} d^4 x_1 \cdots \int_{(x \circ x_1 \circ \cdots \circ x_m)}^{\sigma(x_{m-1})} d^4 x_m \, [L(x), L(x'), L(x_1), \ldots, L(x_m)] \tag{I·3}$$

holds for every m; because we have then

$$i\hbar \frac{\delta H_m(x, n(x))}{\delta \sigma(x')} = -\frac{1}{2} \frac{1}{2^{m-1}} \left(\frac{-i}{\hbar} \right)^{m-1} K_{m-1}, \tag{I·4}$$

which leads, when summed up with respect to m, to

$$i\hbar \frac{\delta H(x, n(x))}{\delta \sigma(x')} = -\frac{1}{2} [H(x, n(x)), H(x', n(x'))], \tag{I·5}$$

and then to (2·1).

Now the relation (I·3) is obviously valid for $m=0$, and also for $m=1$, as is easily seen by applying Jacobi's identity.

$$K_1 = \int_{(x \circ x_1)}^{\sigma(x)} d^4 x_1 \{ [L(x), L(x_1), L(x')] + [L(x), [L(x'), L(x_1)]] \}$$

$$= \int_{(x \circ x' \circ x_1)}^{\sigma(x)} d^4 x_1 [L(x), L(x'), L(x_1)]. \tag{I·6}$$

Assuming that (I·3) is correct for $m=p-1$, one can in general verify that it also holds for $m=p$. But to avoid pompous notations which we need for a general case, we shall illustrate the procedure of mathematical induction by an example $p=4$, that is to say the step from $m=3$ to $m=4$.

Writing x instead of $L(x)$, x' instead of $L(x')$, 1 instead of $L(x_1)$, 2 instead of $L(x_2)$, and so on, we have

$$K_3 = \int^{\sigma(x)} d^4 x_1 \int^{\sigma_1} d^4 x_2 \int^{\sigma_2} d^4 x_3 \, [x, 1, 2, 3, x']$$

$$+ \int^{\sigma(x)} d^4 x_1 \int^{\sigma_1} d^4 x_2 \int^{\sigma(x)} d^4 x_3 \, [x, 1, 2, [x', 3]] +$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma(x)} d^4x_2 \int^{\sigma_2} d^4x_3\ [x', 1, [x', 2, 3]]$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3\ [x, [x', 1, 2, 3]]. \tag{I·7}$$

The second line can be transformed into

$$\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3\ [x, 1, 2, [x', 3]]$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_3 \int^{\sigma_3} d^4x_2\ [x, 1, 2, [x', 3]]$$

$$+\int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d^4x_1 \int^{\sigma_1} d^4x_2\ [x, 1, 2, [x', 3]]$$

$$=\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3\ \{[x, 1, 2, [x', 3]]+[x, 1, 3, [x', 2]]+[x, 2, 3, [x', 1]]\}.$$

In the same way the third line of (I·7) becomes

$$\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x^2 \int^{\sigma_2} d^4x_3\ \{[x, 1, [x', 2, 3]]+[x, 2, [x', 1, 3]]+[x, 3[x', 1, 2]]\}.$$

Thus all the terms of (3·8) can be put under common integral sign.

$$\mathrm{K}_3=\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3\ \{[x, 1, 2, 3, x']+[x, 1, 2, [x', 3]]$$

$$+[x, 1, 3, [x', 2]]+[x, 2, 3, [x', 1]]+[x, 1, [x', 2, 3]]$$

$$+[x, 2, [x', 1, 3]]+[x, 3, [x', 1, 2]]+[x, [x', 1, 2, 3]]\}. \tag{I·8}$$

In this way the assumption, that (I·3) is valid for $m=3$, is reduced to the relation that the expression in the curly bracket above is equal to a simple expression:

$$\{\qquad\}=[x, x', 1, 2, 3] \tag{I·9}$$

Now K_4 has the form

$$K_4=\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4\ [x, 1, 2, 3, 4, x']$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma(x)} d^4x_4\ [x, 1, 2, 3, [x', 4]]$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d_4x_4\ [x, 1, 2, [x', 3, 4]]+$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma(x)} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4 \, [x, 1, [x', 2, 3, 4]]$$

$$+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4 \, [x, [x', 1, 2, 3, 4]]. \tag{I·10}$$

to which a similar transformation can be applied. Making use of the relations such as

$$\begin{aligned}\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d^4x_4 &= \int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4 \\ &+\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_3 \int^{\sigma_3} d^4x_2 \int^{\sigma_2} d^4x_4 + \int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_3 \int^{\sigma_3} d^4x_4 \int^{\sigma_4} d^4x_2 \\ &+\int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_4 + \int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d^4x_1 \int^{\sigma_1} d^4x_4 \int^{\sigma_4} d^4x_2 \\ &+\int^{\sigma(x)} d^4x_3 \int^{\sigma_3} d^4x_4 \int^{\sigma_4} d^4x_1 \int^{\sigma_1} d^4x_2,\end{aligned} \tag{I·11}$$

we obtain

$$\begin{aligned}K_4 = \int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4 \, \{&[x, 1, 2, 3, 4, x'] + [x, 1, 2, 3, [x', 4]] \\ &+[x, 1, 2, 4, [x', 3]] + [x, 1, 3, 4, [x', 2]] + [x, 2, 3, 4, [x', 1]] \\ &+[x, 1, 2, [x', 3, 4]] + [x, 1, 3, [x', 2, 4]] + [x, 1, 4, [x', 2, 3]] \\ &+[x, 2, 3, [x', 1, 4]] + [x, 2, 4, [x', 1, 3]] + [x, 3, 4, [x', 1, 2]] \\ &+[x, 1, [x', 2, 3, 4]] + [x, 2, [x', 1, 3, 4]] + [x, 3, [x', 1, 2, 4]] \\ &+[x, 4, [x', 1, 2, 3]] + [x, [x', 1, 2, 3, 4]]\}.\end{aligned} \tag{I·12}$$

These brackets can be so combined into pairs that they are at once reduced to single brackets.

$$\left.\begin{aligned}&[x, 1, 2, 3, 4, x'] + [x, 1, 2, 3, [x', 4]] = [x, 1, 2, 3, x', 4], \\ &[x, 1, 2, 4, [x', 3]] + [x, 1, 2, [x', 3, 4]] = [x, 1, 2, [x', 3], 4], \\ &[x, 1, 3, 4, [x', 2]] + [x, 1, 3, [x', 2, 4]] = [x, 1, 3, [x', 2], 4], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &[x, 4, [x', 1, 2, 3]] + [x, [x', 1, 2, 3, 4]] = [x, [x', 1, 2, 3], 4].\end{aligned}\right\} \tag{I·13}$$

Thus the sum of all these takes the form

$$[\{[x, 1, 2, 3, x'] + [x, 1, 2, [x', 3]] + \ldots\ldots + [x, [x', 1, 2, 3]]\}, 4], \tag{I·14}$$

but the expression { } is just the left hand side of (I·9) and so we obtain

$$K_4=\int^{\sigma(x)} d^4x_1 \int^{\sigma_1} d^4x_2 \int^{\sigma_2} d^4x_3 \int^{\sigma_3} d^4x_4\ [L(x), L(x'), L(x_1), L(x_2), L(x_3), L(x_4)], \tag{I·15}$$

what was to be verified.

Appendix II.

An example of non-local interaction.

Let us assume the action function for a system of interacting electronic and electromagnetic fields in the following form:

$$\begin{aligned} \mathrm{I}=&\int L_{\text{radiation}}(x)d^4x+\int L_{\text{electron}}(\xi)d^4\xi \\ &+ie\iint A_\mu(x)\,\bar{\psi}(\xi)\gamma^\mu\psi(\xi)\ \eta(x-\xi)\ d^4x d^4\xi. \end{aligned} \tag{II·1}$$

$\eta(x-\xi)$ in the interaction term is a scalar function which represents in a certain sense the extension of the electron's charge.

The equations of the fields formally derived from this Lagrangian are

$$\square^2 A_\mu(x)=ie\int\bar{\psi}(\xi)\gamma_\mu\psi(\xi)\,\eta(x-\xi)\ d^4\xi,$$

$$\left\{\gamma_\mu\left(\frac{\partial}{\partial\xi_\mu}-ie\int A^\mu(x)\,\eta(x-\xi)\ d^4x\right)+\varkappa\right\}\psi(\xi)=0, \tag{II·2}$$

$$\left\{\tilde{\gamma}_\mu\left(\frac{\partial}{\partial\xi_\mu}+ie\int A^\mu(x)\,\eta(x-\xi)\ d^4x\right)-\varkappa\right\}\bar{\psi}(\xi)=0. \tag{II·3}$$

They may be interpreted as describing the interaction through extended charge.

The gauge transformation is carried out as follows:

$$A_\mu(x)\rightarrow A_\mu(x)-\frac{\partial\Lambda(x)}{\partial x^\mu},\quad \square^2\Lambda(x)=0\,; \tag{II·4}$$

$$\left.\begin{aligned} &\psi(\xi)\rightarrow \exp\left(-ie\int\Lambda(x)\,\eta(x-\xi)\ d^4x\right)\psi(\xi), \\ &\bar{\psi}(\xi)\rightarrow\bar{\psi}(\xi)\ \exp\left(ie\int\Lambda(x)\,\eta(x-\xi)\,d^4x\right)\ ; \end{aligned}\right\} \tag{II·5}$$

where $\lim\limits_{x\rightarrow\infty}\eta(x)\rightarrow 0$ is assumed.

Now we apply the integral formulation described in § 6 to this Lagrange function, and the S-matrix in the interaction representation is given by

$$S=\exp\frac{ie}{\hbar}\int_{-\infty}^{\infty}d^4x_1\int_{-\infty}^{+\infty}d^4x_2\bar{\psi}(x_2)\gamma_\mu\psi(x_2)\,\eta(x_2-x_1)\,A^\mu(x_1). \tag{II·6}$$

The exponential function is interpreted according to (6·2). We shall study the self-energy of an electron in e^2-approximation and so pick up the element of S-matrix corresponding to the transition (1 *electron, no photon*) → (1 *electron, no photon*). This is

$$\left(\frac{ie}{\hbar}\right)^2\iiiint_{-\infty}^{+\infty} d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, \bar{\psi}_\omega(x_2)\,(\gamma_\mu)_{\omega\tau}$$
$$\times \langle P^*(\psi_\tau(x_2)\cdot\bar{\psi}_\sigma(x_1))\rangle_{\rm vac}\,(\gamma_\nu)_{\sigma\rho}\,\psi_\rho(x_1)$$
$$\times \langle P^*(A^\nu(x_3),\, A^\mu(x_4))\rangle_{\rm vac}\,\eta(x_1-x_3)\,\eta(x_2-x_4). \qquad \text{(II·7)}$$

The Feynman diagram for this transition is shown in fig. 7. One can see that the electron undergoes deflection at points x_1 and x_2, while the virtual photon appears and vanishes at x_3 and x_4. The function η acts as an intermediary between these points and conveys energy and momentum*.

We put

$$\eta(x)=\frac{1}{(2\pi)^4}\int e^{-i(\boldsymbol{l},x)}\,\xi(\boldsymbol{l})\,d^4\boldsymbol{l}; \qquad \text{(II·8)}$$

then the conservation of energy and momentum requires that the Fourier component of η, which connects 1 and 3, or 2 and 4 must be equal to the energy-momentum of the virtual photon $\boldsymbol{k}$.

Let us assume in particular that $\xi(\boldsymbol{l})$ is given by

$$\xi(\boldsymbol{l})=\frac{M}{\sqrt{\boldsymbol{l}^2+M^2}}, \qquad \text{(II·9)}$$

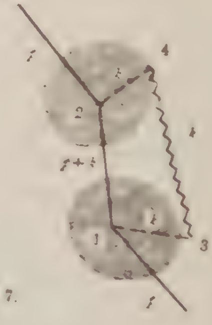

Fig. 7.

where M is of the order $\hbar$ divided by the classical electron radius, that is to say the Compton frequency of a meson. (In the limit $M\to\infty$, $\xi(\boldsymbol{l})$ becomes unity and $\eta(x)$ is reduced to the δ-function, and the whole scheme coincides with the ordinary one.) The presence of the η-function then gives rise to a change in the integrand, namely the factor $\frac{1}{\boldsymbol{k}^2}$ of the usual theory ($\boldsymbol{k}$ being the energy-momentum of the virtual photon) is replaced by

$$\frac{1}{\boldsymbol{k}^2}\to\frac{1}{\boldsymbol{k}^2}\cdot\frac{M^2}{\boldsymbol{k}^2+M^2}=\frac{1}{\boldsymbol{k}^2}-\frac{1}{\boldsymbol{k}^2+M^2}. \qquad \text{(II·10)}$$

Thus the assumption (II·9) has the same effect with Feynman's cut off factor;[6)] consequently it can make the self-energy of an electron finite, but is incompetent as to the problem of the vacuum polarization.

As is easily seen from (II·10), such an extension function η in the Lagrangian function yields the difficulty of negative energy when one goes over to the Hamilton formalism,** but we shall not enter into details about this model, since it has been mentioned merely as an example.

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* Of course all this description has only a metaphorical meaning.

** Also the consistency of (II·2)—(II·5) with (2·6) should be further investigated.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Canonical Theory of Quantum Electrodynamics

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(Received August 7, 1950)

Introduction

In applying the general theory of quantized wave fields due to Heisenberg and Pauli to the electromagnetic field, one encounters with the notorious difficulty that the canonical conjugate of the scalar potential identically vanishes.

There have been proposed several ways out of this difficulty, but all of them spoil, it seems, more or less the elegance of the general theory. Rosenfeld[1] has indicated how to construct a theory, invariant under both Lorentz and gauge transformations. But his method of formulation seems unsatisfactory to the authors. According to his theory the time derivative of the scalar potential Φ, is put equal to an arbitrary *c*-number function $A(x)$. But this assumption is not Lorentz-invariant. The postulates of Lorentz and gauge invariance seem necessarily to result in a definite interpretation of $\dot{\Phi}$.

One of the present authors has examined to remove this inconsistency and to formulate the covariant formulation of the quantum electrodynamics along the line of reasoning of Tomonaga and Schwinger. The contents of this investigation will be published elsewhere.(2)

In the present paper we shall make some investigations about the modified Maxwell equations obtained from the above mentioned reasoning. Finally, similar considerations will also be applicable to the (pseudo-) vector meson fields.

§ 1. Equations of the pure electromagnetic field.

Adopting the classical Lagrangian density

$$L_f=\frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{H}^2)^{(3)}, \tag{1}$$

where

$$\boldsymbol{H}=\mathrm{rot}\,\boldsymbol{A},$$

$$\boldsymbol{E}=\mathrm{grad}\,\Phi+\frac{\partial \boldsymbol{A}}{\partial t}^{(4)},$$

the canonical conjugate of $\boldsymbol{A}$ is

$$\boldsymbol{E}=\frac{\partial L_f}{\partial \dot{\boldsymbol{A}}} \tag{2}$$

and that of Φ is

$$\frac{\partial L_f}{\partial \dot{\Phi}}=0. \tag{3}$$

The vanishing of the last quantity interferes with the construction of canonical formalism.

To remove this difficulty, Rosenfeld has introduced a new quantity F which is canonically conjugate to Φ and is assumed to be a world scalar.

Now the eq. (2) enables one to calculate $\dot{\boldsymbol{A}}$ as the functions of $\boldsymbol{E}$ and Φ, but $\dot{\Phi}$ remains completely undeterminate.
That is

$$\dot{\Phi}=\Lambda(x)=\text{arbitrary}. \tag{4}$$

According to the general scheme of Heisenberg and Pauli, the commutation relations ([C. R]) of the field quantities are given by

$$\begin{aligned}&[E_\mu(x,t), A_\nu(x',t)]=-i\delta_{\mu\nu}\delta(x-x')\\ &\qquad\qquad\qquad \mu,\nu=1,2,3,\\ &[F(x,t), \Phi(x',t)]=-i\delta(x-x').\end{aligned} \tag{5}$$

All the other combinations not presented here are zero.

The Hamiltonian of the system considered is

$$\overline{\mathfrak{H}}_f=\int\mathfrak{H}_f dx$$

$$\mathfrak{H}_f=\frac{1}{2}(\boldsymbol{E})^2-\boldsymbol{E}\,\text{grad}\,\Phi+F\Lambda+\frac{1}{2}(\boldsymbol{H})^2 \tag{6}$$

and the field equations are

$$\frac{d\theta}{dt}=i[\overline{\mathfrak{H}}_f, \theta], \tag{7}$$

where θ is any function of field quantities.
That is

$$\dot{\boldsymbol{A}}=\boldsymbol{E}-\text{grad}\,\Phi, \tag{7.1}$$

$$\dot{\boldsymbol{E}}=\Delta\boldsymbol{A}-\text{grad}(\text{div}\,\boldsymbol{A}), \tag{7.2}$$

$$\dot{F}=-\text{div}\,\boldsymbol{E}, \tag{7.3}$$

$$\dot{\Phi}=\Lambda \tag{7.4}$$

provided that Λ is an arbitrary *c*-number function.

If we eliminate $\boldsymbol{E}$ from (7.1), (7.2) and (7.3) and make use of the condition $F\Psi_0=0$ then we obtain the equations which are completely the same ones as the Eulerian equations derived from (1). Therefore (7.1)→(7.3) are practically Lorentz covariant. But the last eq. (7.4), which stands alone, violate Lorentz invariance of the whole scheme. If we replace Λ by an adequate q-number quantity, then the whole system of equations may become invariant under Lorentz transformations.

Now we propose

$$\dot{\Phi}=-\mathrm{div}\boldsymbol{A}+\Lambda'(x) \tag{7.4$'$}$$

where Λ' is any arbitrary c-number function. However, the function Λ' is not of any importance. Since this function can be eliminated by a well known gauge transformation, we can put Λ' to be zero without loss of generality.

Then the field eqs. (7.1) (7.3) remain to be valid but instead of (7.2) we obtain

$$\dot{\boldsymbol{E}}=\Delta\boldsymbol{A}-\mathrm{grad}(\mathrm{div}\,\boldsymbol{A})-\mathrm{grad}\,F. \tag{7.2$'$}$$

From (7.1), (7.2)′, (7.3) and (7.4)′, we obtain

$$\Box\boldsymbol{E}=0, \qquad \Box F=0, \qquad \Box\boldsymbol{A}=\mathrm{grad}\,F, \qquad \Box\Phi=-\dot{F}. \tag{8}$$

Therefore, if we assume

$$F\Psi_0=0, \qquad \dot{F}\Psi_0=0 \tag{9}$$

at $t=0$, (Ψ_0 is the state-vector in the Heisenberg representation) then it holds

$$F\Psi_0=0$$

at any instant on account of (8).
The second condition of (9) can be replaced by

$$\mathrm{div}\,\boldsymbol{E}\Psi_0=0. \tag{9$'$}$$

From (8), the behaviour of the field quantities at any world point can be expressed by those at $t=0$. For example,

$$\boldsymbol{E}(x,t)=\int_{t'=0}\Big[D(x-x',t)\{\Delta'\boldsymbol{A}(x',0)-\mathrm{grad}'(\mathrm{div}'\boldsymbol{A}(x',0))-\mathrm{grad}'F(x',0)\}$$
$$+\frac{\partial D(x-x',t)}{\partial t}\boldsymbol{E}(x',0)\Big]dx',$$

$$F(x,t)=\int_{t'=0}\left\{-D(x-x',t)\mathrm{div}'\boldsymbol{E}(x',0)+\frac{\partial D(x-x',t)}{\partial t}F(x',0)\right\}dx'. \tag{10}$$

[C. R] of $\boldsymbol{A}$, Φ at any two world points can also be obtained from (8) and (5) and are expressed as follows;

$$[A_\mu(x,t),A_\nu(x',t')]=-ig_{\mu\nu}D(x-x',t-t')$$
$$+i\int_{t'}^{t}\frac{\partial D(X-X'')}{\partial x^{\mu''}}\frac{\partial D(X''-X')}{\partial x^{\nu''}}(dx'')^4\quad(\mu,\nu=0,1,2,3),\tag{11}$$

where $$\boldsymbol{A}=(A_1\,A_3\,A_3),\qquad A_0=-\Phi$$

and $g_{\mu\nu}$ is the usual metric tensor

$$g_{11}=g_{22}=g_{33}=-g_{00}=1.$$

(11) is Lorentz invariant as easily seen.

Finally, it should be noticed that, for the gauge transformation

$$\boldsymbol{A}\to\boldsymbol{A}'=\boldsymbol{A}+\operatorname{grad}\lambda,$$
$$\Phi\to\Phi'=\Phi-\dot{\lambda},\tag{12}$$

λ must satisfy the wave equation

$$\Box\lambda=0\tag{13}$$

by virtue of the gauge invariance of (7.4)′.

§ 2. Electrons and electromagnetic fields.

(Interaction representation)

Let us consider the system of electrons interacting with electromagnetic field. The Hamiltonian density of the total system is

$$\mathfrak{H}=\mathfrak{H}_0+V,\qquad V=-\boldsymbol{j}\boldsymbol{A}+\rho\Phi,$$
$$\mathfrak{H}_0=\mathfrak{H}_f+\mathfrak{H}_m,\qquad \mathfrak{H}_m=\psi^*\frac{1}{i}\boldsymbol{\alpha}\operatorname{grad}\psi+\varkappa\psi^*\beta\psi,\tag{14}$$
$$\boldsymbol{j}=e\psi^*\boldsymbol{\alpha}\psi,\qquad \rho=e\psi^*\psi.$$

[C. R] of electron field is

$$\{\psi^*_\alpha(x,t),\psi_\beta(x',t)\}=\delta_{\alpha\beta}\delta(x-x').\tag{15}$$

Now, the equations of the electromagnetic field are

$$\dot{\boldsymbol{A}}=\boldsymbol{E}-\operatorname{grad}\Phi,\qquad \dot{\Phi}=-\operatorname{div}\boldsymbol{A},$$
$$\dot{\boldsymbol{E}}=\Delta\boldsymbol{A}-\operatorname{grad}(\operatorname{div}\boldsymbol{A})-\operatorname{grad}F+\boldsymbol{j},\tag{16}$$
$$\dot{F}=-\operatorname{div}\boldsymbol{E}-\rho.$$

And the supplementary conditions (9) (9)′, now should be replaced by

$$F\Psi_0=0,\qquad (\operatorname{div}\boldsymbol{E}+\rho)\Psi_0=0.\tag{17}$$

The proof of the Lorentz covariance of (9) or (17) will be given in (11).

Now let us consider the gauge transformation

$$\begin{aligned} &\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \operatorname{grad} \lambda, \\ &\boldsymbol{\Phi} \rightarrow \boldsymbol{\Phi}' = \boldsymbol{\Phi} - \dot{\lambda}, \\ &\psi \rightarrow \psi' = e^{i\varepsilon\lambda}\psi, \\ &\psi^* \rightarrow \psi^{*\prime} = e^{-i\varepsilon\lambda}\psi^*. \end{aligned} \tag{18}$$

The proof of the invariance of field eqs. for the transformation (18) can be easily given. The infinitesimal generating operator G for (18) is

$$G = -\int \{ (\operatorname{div} \mathbf{E} + \rho)\lambda + F\dot{\lambda} \} \, d\boldsymbol{x}, \tag{19}$$

because

$$\delta \mathbf{A} = i[G, \mathbf{A}] \text{ etc.}$$

hold. Using field eqs. and (13) we can easily show that G is a constant of motion, that is

$$\frac{dG}{dt} = 0. \tag{20}$$

So far, we have made use of the Heisenberg representation. Hereafter we shall work on the interaction representation. This transformation of representation can be performed as follows:

$$\mathbf{A} \rightarrow A = U\mathbf{A}U^{-1}, \quad \boldsymbol{\Psi}_0 \rightarrow \Psi(t) = U(t)\boldsymbol{\Psi}_0 \tag{21}$$

(any quantity in the Heisenberg representation will be expressed by bold face letters), where U is a unitary operator defined by the equation

$$i\frac{dU}{dt} = \overline{V}U, \qquad \overline{V} = U\overline{\mathbf{V}}U^{-1}.$$

The equation of motion of any quantity A now takes the form

$$\frac{dA}{dt} = i[\mathfrak{H}_0, A],$$

where the interaction term disappears, while the Schroedinger equation for $\Psi(t)$ runs

$$i\frac{d\Psi}{dt} = \overline{V}\Psi. \tag{22}$$

But the supplementary conditions retain the original form;

$$F\Psi(t) = 0, \qquad (\operatorname{div} \mathbf{E} + \rho)\Psi(t) = 0, \tag{23}$$

where the instant t at which F and $K = \operatorname{div} \mathbf{E} + \rho$ are considered coincides with the time of $\Psi(t)$. But if the former differs from the latter, (23) should be replaced by

$$\begin{aligned}&\left\{F(x,t)-\int_{t'=\text{const.}} D(x-x',t-t')\,\rho(x',t')\,dx'\right\}\Psi(t')=0,\\&\left\{\operatorname{div}\boldsymbol{E}(x,t)+\int_{t'=\text{const.}}\frac{\partial D(x-x',t-t')}{\partial t}\rho(x',t')\,dx'\right\}\Psi(t')=0.\end{aligned}\tag{23}'$$

(23)′ can easily be obtained if we use (23) and (10). The equation

$$\frac{d\boldsymbol{G}}{dt}=0 \tag{20}$$

now is transformed into

$$\frac{dG}{dt}=\frac{d}{dt}(U\boldsymbol{G}U^{-1})=-i[\overline{V},G],$$

that is

$$\frac{dG}{dt}+i[\overline{V},G]=0. \tag{24}$$

Substituting from (19) into (24) and taking notice of λ and $\dot{\lambda}$ being arbitrary, the following equations can be derived;

$$\begin{aligned}\frac{dK}{dt}&=-\Delta F-i[\overline{V},K],\\\frac{dF}{dt}&=-K-i[\overline{V},F].\end{aligned}\tag{25}$$

By means of (25), we can show the compatibility of eq. (22) and conditions (23) (or (23)′).

§ 3. Elimination of the longitudinal components.

Let us separate the vector potential $\boldsymbol{A}$ into two parts in the following way

$$\boldsymbol{A}=\boldsymbol{B}-\operatorname{grad}\Gamma, \tag{26}$$

where we postulate

$$\operatorname{div}\boldsymbol{B}=0. \tag{26}'$$

Hence $\boldsymbol{B}$ is the transversal vector potential. Substituting from (26) into (7.4)′ ($A'=0$) the following eq. is derived;

$$\frac{d\Phi}{dt}=\Delta\Gamma.$$

If we make use of a Green function $G(x)$, which is defined by

$$\Delta G(x)=\delta(x)\qquad G(x)=G(-x),$$

then

$$\Gamma(x,t)=\int G(x-x')\,\dot{\Phi}(x',t)\,dx'=-\int G(x-x')\,\mathrm{div}'\,\boldsymbol{A}(x',t)\,dx'. \tag{27}$$

By virtue of this solution (27) and the [C. R] (5), it holds

$$[\boldsymbol{E}(x,t),\,\Gamma(x',t)]=-i\,\mathrm{grad}\,G(x-x') \tag{28}$$

and further

$$[\mathrm{div}\,\boldsymbol{E}(x,t),\,\Gamma(x',t)]=-i\,\delta(x-x'). \tag*{(28)$'$}$$

That is, div $\boldsymbol{E}$ is the canonical conjugate quantity of Γ.

Commutators of Γ and the other quantities turn out to be zero.

As for $\boldsymbol{B}$, it is governed by the eq.

$$\Box\,\boldsymbol{B}=0 \tag{29}$$

and [C. R]'s are given by

$$[E_\mu(x),B_\nu(x')]_t=-i\,\delta_{\mu\nu}\delta(x-x')+i\frac{\partial^2 G(x-x')}{\partial x^\mu\partial x^\nu}$$

$$\mu,\nu=1,2,3,$$

$$[\mathrm{div}\,\boldsymbol{E}(x),\,\boldsymbol{B}(x')]_t=0, \tag{30}$$

$$[F,\boldsymbol{B}]=[\Gamma,\boldsymbol{B}]=0.$$

Now let us consider the following unitary transformation:

$$\Psi(t)\rightarrow\Psi'(t)=e^{iR(t)}\Psi(t)$$

where

$$R(t)=\int\rho(x,t)\,\Gamma(x,t)\,dx.$$

Then the condition

$$K\Psi(t)=0,\qquad K=\mathrm{div}\,\boldsymbol{E}+\rho$$

becomes

$$e^{iR}K\,e^{-iR}\Psi'(t)=0.$$

Making use of the [C. R] (28), we can calculate $e^{iR}K\,e^{-iR}$, the result of which is

$$e^{iR}K\,e^{-iR}=\mathrm{div}\,\boldsymbol{E}.$$

Hence the second condition of (23) takes the form

$$\mathrm{div}\,\boldsymbol{E}\,\Psi'=0. \tag{31}$$

On the other hand the Schroedinger eq. (22) becomes

$$i\frac{\partial\Psi'}{\partial t}=\overline{V}\Psi',$$

where

$$\overline{V}'=e^{iR}\overline{V}e^{-iR}+i\frac{de^{iR}}{dt}e^{-iR}$$

$$=-\int \boldsymbol{jB}\,d\boldsymbol{x}-\frac{1}{2}\iint\rho(x)\,G(x-x')\,\rho(x')\,d\boldsymbol{x}\,d\boldsymbol{x}'$$

$$+\iint\rho(x')\,G(x'-x)\,\mathrm{div}\,\boldsymbol{E}\,d\boldsymbol{x}'\,d\boldsymbol{x}.$$

Making $\overline{V}'$ operate on Ψ' and taking into account of (31), we obtain

$$i\frac{\partial\Psi'}{\partial t}=\overline{V}''\Psi', \tag{32}$$

$$\overline{V}''=-\int\boldsymbol{jB}\,d\boldsymbol{x}-\frac{1}{2}\iint\rho\,G(x-x')\,\rho'\,d\boldsymbol{x}\,d\boldsymbol{x}'. \tag{33}$$

This is completely identical with the equation given by Schwinger.

Finally the generating operator G of the gauge transformation is transformed as follows:

$$G\rightarrow G'=e^{iR}Ge^{-iR}$$

$$=-\int\{F\dot{\lambda}+(\mathrm{div}\,\boldsymbol{E})\lambda\}d\boldsymbol{x}.$$

This is the generating operator in case of the free electromagnetic field. Taking notice of [C. R] (30), $\boldsymbol{B}$ is gauge invariant and hence the modified Schroedinger eq. (32) is also invariant.

§ 4. Fourier expansion.

In order to apply our formalism to practical problems, it is useful to expand the field quantities into Fourier integrals.

Let us put

$$\begin{aligned}
\boldsymbol{E}&=\frac{1}{\sqrt{2\pi}^{3}}\int \boldsymbol{e}(k,t)e^{i\boldsymbol{kx}}d\boldsymbol{k},\\
F&=\frac{1}{\sqrt{2\pi}^{3}}\int f(k,t)e^{i\boldsymbol{kx}}d\boldsymbol{k},\\
\boldsymbol{A}&=\frac{1}{\sqrt{2\pi}^{3}}\int \boldsymbol{u}(k,t)e^{i\boldsymbol{kx}}d\boldsymbol{k},\\
\boldsymbol{B}&=\frac{1}{\sqrt{2\pi}^{3}}\int \beta(k,t)e^{i\boldsymbol{kx}}d\boldsymbol{k},
\end{aligned} \tag{34}$$

$$\Phi=\frac{1}{\sqrt{2\pi}^{3}}\int v(k,t)e^{i\boldsymbol{k}x}d\boldsymbol{k}.$$

Substituting from (34) into field equations, we obtain

$$\begin{aligned}
\boldsymbol{u}(k,t)&=\boldsymbol{a}(k)e^{-ikt}+\boldsymbol{a}^{*}(-k)e^{ikt}-i\frac{\boldsymbol{k}}{k}t\{g(k)e^{-ikt}+g^{*}(-k)e^{ikt}\},\\
v(k,t)&=-\{a_0(k)e^{-ikt}+a_0^{*}(-k)e^{ikt}\}-it\{g(k)e^{-ikt}-g^{*}(-k)e^{ikt}\},\\
\boldsymbol{e}(k,t)&=-i\left[\left\{\boldsymbol{c}(k)+\frac{\boldsymbol{k}}{k}g(k)\right\}e^{-ikt}-\left\{\boldsymbol{c}^{*}(-k)-\frac{\boldsymbol{k}}{k}g^{*}(-k)\right\}e^{ikt}\right],\\
f(k,t)&=-2i\{g(k)e^{-ikt}-g^{*}(-k)e^{ikt}\},\\
\boldsymbol{\beta}(k,t)&=\sum_{\lambda=1,2}\{\beta_\lambda(k)\boldsymbol{\alpha}_\lambda(k)e^{-ikt}+\beta_\lambda^{*}(-k)\boldsymbol{\alpha}_\lambda(-k)e^{ikt}\},
\end{aligned}\tag{35}$$

$$\boldsymbol{c}(k)=k\boldsymbol{a}(k)+\boldsymbol{k}a_0(k),$$

$$g(k)=ka_0(k)+(\boldsymbol{k}\boldsymbol{a}(k)).$$

(the definition of β_λ $\boldsymbol{\alpha}_\lambda$ will be given below).

Using (35) the Hamiltonian of the electromagnetic field can be written

$$\begin{aligned}
\overline{\mathfrak{H}}_f=\int k^2[&\boldsymbol{a}^{*}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{a}^{*}+(\boldsymbol{k}\boldsymbol{a}^{*})(\boldsymbol{k}\boldsymbol{a})+(\boldsymbol{k}\boldsymbol{a})(\boldsymbol{k}\boldsymbol{a}^{*})\\
&-2(a_0^{*}a_0+a_0a_0^{*})+2(g^{*}g+gg^{*})]d\boldsymbol{k}.
\end{aligned}\tag{36}$$

The total momentum of the field is

$$\begin{aligned}
\boldsymbol{P}_f&=\int\{(\boldsymbol{E}\operatorname{grad}\boldsymbol{A})+F\operatorname{grad}\Phi\}\,dx\\
&=-\int\frac{\boldsymbol{k}}{k}[k^2\{(\boldsymbol{a}^{*}\boldsymbol{a})+(\boldsymbol{a}\boldsymbol{a}^{*})-2(a_0^{*}a_0+a_0a_0^{*})\}\\
&\qquad+(\boldsymbol{k}\boldsymbol{a}^{*})(\boldsymbol{k}\boldsymbol{a})+(\boldsymbol{k}\boldsymbol{a})(\boldsymbol{k}\boldsymbol{a}^{*})]d\boldsymbol{k}.
\end{aligned}\tag{37}$$

The [C. R]'s of a's are

$$\begin{aligned}
&[a_\mu(k),a_\nu^{*}(k')]=\frac{1}{2}\left(\frac{\delta_{\mu\nu}}{k}-\frac{1}{2}\frac{k_\mu k_\nu}{k^3}\right)\delta(\boldsymbol{k}-\boldsymbol{k}'),\quad \mu,\nu=1,2,3\\
&[a_\mu(k),a_0^{*}(k')]=[a_0(k),a_\mu^{*}(k')]=0,\\
&[a_0(k),a_0^{*}(k')]=-\frac{1}{4k}\delta(\boldsymbol{k}-\boldsymbol{k}').
\end{aligned}\tag{38}$$

Now putting

$$\boldsymbol{a}(k)=\sum_{\lambda=1,2,3}\boldsymbol{\alpha}_\lambda(k)\beta_\lambda(k),\quad(\boldsymbol{\alpha}_\mu(k)\boldsymbol{\alpha}_\nu(k))=\delta_{\mu\nu},$$

$$\boldsymbol{\alpha}_3(k)=\boldsymbol{k}/k,\qquad \mu,\nu,\lambda=1,2,3$$

we have

$$[\beta_\lambda(k),\beta_\mu^{*}(k')]=\frac{1}{2k}\delta_{\lambda\mu}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\qquad \lambda,\mu=1,2,$$

$$[\beta_3(k), \beta_3^*(k')] = \frac{1}{4k}\delta(\boldsymbol{k}-\boldsymbol{k}').$$

Other combinations vanish.

Furthermore let us introduce the following new quantities:

$$\sqrt{2k}\,\beta_\lambda(k) = b_\lambda(k), \quad 2\sqrt{k}\,\beta_3(k) = b_3(k), \quad 2\sqrt{k}\,a_0(k) = b_0(k),$$
$$\lambda = 1, 2$$

then we have

$$[b_\lambda(k), b_\mu^*(k')] = \delta_{\lambda\mu}\delta(\boldsymbol{k}-\boldsymbol{k}'),$$
$$[b_0(k), b_0{}^*(k')] = -\delta(\boldsymbol{k}-\boldsymbol{k}').$$

Let us define

$$b_\lambda^*(k)b_\lambda(k) = n_\lambda(k),$$
$$(\lambda = 1, 2, 3)$$
$$b_0(k)b_0{}^*(k) = n_0(k)$$

where all the n's take on zero or positive integral values only. Computations of $\overline{\mathfrak{H}}_f$ and $\boldsymbol{P}_f$ result in

$$\overline{\mathfrak{H}}_f = \sum_k k(n_1(k) + n_2(k) + n_3(k) - n_0(k) + 1)$$
$$+ 2\sum_k \{g^*(k)g(k) + g(k)g^*(k)\}, \tag{39}$$

$$\boldsymbol{P}_f = -\sum_k \boldsymbol{k}(n_1(k) + n_2(k) + n_3(k) - n_0(k)). \tag{39}$$

Now the conditions

$$F\Psi' = 0, \qquad \operatorname{div} \boldsymbol{E}\Psi' = 0$$

run in this Fourier representation as follows:

$$g\Psi' = 0, \qquad g^*\Psi' = 0. \tag{40}$$

Now we try to find such a state which is an eigen-state of (39) and (39)′ and at the same time satisfies the conditions (40). Since the pair (b_1, b_2) and the pair (b_3, b_0) are mutually commutative we may confine our attention to the latter pair only. For the sake of simplicity, we shall write n and m as the independent variables instead of n_0 and n_3 and further $C(n, m)$ for the components of Ψ'.

Then our objective is to determine the vector $C(n, m)$ in the Hilbert space which satisfies the following eigenvalue problems:

$$(\boldsymbol{n}_0 - \boldsymbol{n}_3)\Psi' = \mu\Psi', \tag{41}$$

$$g\Psi' = \lambda\Psi', \tag{42}$$

$$g^*\Psi' = \lambda^*\Psi'. \tag{43}$$

The matrix elements of $\boldsymbol{n}_0-\boldsymbol{n}_3$, g, and g^* are

$$(nm|\boldsymbol{n}_0-\boldsymbol{n}_3|n'm')=(n-m)\delta_{nn'}\delta_{mm'}\,,$$

$$(nm|g|n'm')=\frac{\sqrt{k}}{2}\{\sqrt{n}\,\delta_{n,n'+1,}\delta_{m,m'}+\sqrt{m+1}\,\delta_{m,m'-1,}\delta_{n,n'}\},$$

$$(nm|g^*|n'm')=(n'm'|g|nm).$$

In the first place from (41), we infer that the only non-vanishing elements of C are

$$C(\mu+l,l) \qquad l=0,1,2\cdots\cdots$$
$$\mu=\text{a constant} \qquad (0,1,2\cdots\cdots). \tag{44}$$

For the moment we shall assume $\lambda\fallingdotseq 0$, and we obtain from (42)

$$\sqrt{n}\;C(n-1,m)+\sqrt{m+1}C(n,m+1)=\frac{2}{\sqrt{k}}\lambda\,C(n,m). \tag{45}$$

If we put in this eq. $n=\mu+1$, $m=0$, then, taking notice of (44), we obtain

$$C(\mu+l,l)=(-1)^l\sqrt{\frac{(\mu+l)!}{\mu!\,l!}}C(\mu,0) \tag{46}$$

where λ is an arbitrary constant.

Next we put $n=\mu$ $m=0$ in (45), then we obtain

$$\lambda C(\mu,0)=0.$$

From this it follows that λ should be zero.

Finally, following completely the same line of reasoning, we obtain the following result from (43);

$$C(\mu+l,l)=(-1)^l\sqrt{\frac{\mu!\,l!}{(\mu+l)!}}C(\mu,0). \tag{47}$$

This must be compared with (46). We can conclude from the above as follows:

If and only if $\lambda=\mu=0$, the solution $C(n,m)$ can exists and its only surviving elements are

$$C(l,l)=(-1)^lC(0,0) \tag{48}$$

the norm of which obviously diverges.

Further we can not define the vacuum state by

$$n_\lambda=0, \qquad \lambda=0,1,2,3$$

but rather by

$$n_1=n_2=0, \quad n_3=n_0 \quad \text{(taking every positive integral value and zero).}$$

This conclusion differs from those defined by Schwinger and if we do postulate $n_3=n_0=0$, then $g\Psi'=0$ would be violated.

§ 5. Conclusion.

We have formulated the quantum electrodynamics following the general scheme of Heisenberg and Pauli not using the Lorentz condition from the beginning, and have shown the equivalence of our formulation with the ordinary one if we eliminate the longitudinal components. But as we see from (35), the Fourier components of $\boldsymbol{A}$, Φ contain terms which are proportional to t. If we make $\boldsymbol{A}$, Φ, operate on Ψ' directly, then these pathological terms disappear by virtue of (40). But if we make the commutators of field quantities operate on Ψ' then these special terms may contribute some effects for aught we know. About these problems, we hope to discuss elsewhere.

References

1) L. Rosenfeld: Ann. d. Phys. **5** (1930), 113.
2) This will be published in near future in this journal. We shall refer this with (II).
3) Natural units are used.
4) The definition of $\boldsymbol{E}$ in this paper has opposite sign as compared with the ordinary definition.

Note added in proof: In the mean time we have found that the whole theory here presented is completely equivalent with the usual one, the proof for which will be given below.

From eq. (16) and [C.R]'s we know that $F(x)$ at any world point commutes with F at every other point x'. Now let us perform the following unitary transformation

$$\Psi_0 \to \Psi_0' = e^{iR}\Psi_0\ , \qquad R = -\frac{1}{2}\int_{-\infty}^{t} F^2 (dx)^4.$$

Then any field quantity A is transformed into

$$A(x) \to A'(x) = e^{iR} A(x) e^{-iR},$$

and the field equation becomes

$$\frac{dA'}{dt} = i\,[\overline{\mathfrak{H}}'', A']\ , \qquad \overline{\mathfrak{H}}'' = e^{iR}\,\overline{\mathfrak{H}} e^{-iR} - i\frac{de^{iR}}{dt}\cdot e^{-iR}.$$

Making use of the character of F stated above, we know that this eq. is nothing but the ordinary one. The conditions (17) now can be written as follows:

$$\left(\frac{\partial A'^{\mu}}{\partial x^{\mu}}\right)\Psi_0' = 0\ , \qquad \frac{d}{dt}\left(\frac{\partial A'^{\mu}}{\partial x^{\mu}}\right)\Psi_0' = 0\ .$$

Q. E. D.

Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

On the Method of the Theory of Elementary Particles*

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(Received August 7, 1950)

1. On the "Elementarity" of Particles.

To determine whether an object is an elementary particle or not, is one of the most interesting problems in the recent years. In some respects this has been a problem pending unsolved ever since the Hellenic age. From a theoretical point of view, to be an elementary particle means that the object in question can be represented by a quantized field. This definition, however, will lose its significance, when the mutual interaction between two such fields is very large. Fortunately or unfortunately, so strong an interaction has never been found to exist.

Among the questions of the elementarities of the particle the most serious one will be that asked about the nucleons. Then, what can be the experimental criterion to judge their elementarity? In this regard, we wish to advocate that the elementarity can be seen in the processes through which the substantial part of the rest mass is created from the energy which is not in the form of rest mass, as it is the case with the pair-creation, provided that there is no substantial effect of the very strong coupling. Let us consider a case where a composite particle composed of elementary particles is created through a process of pair-creation. If the binding energy forming this complex particle is small compared with its rest mass, its creation probability will be far smaller than the creation probability of the constituent elementary particles as separate individuals.

If the nucleon is an elementary particle, there must be negative protons to be observed. If the nucleon is composed of smaller constituents, there can be no negative protons. But if an extremely strong coupling exists, the situation will be quite different, and the very concept of elementary particle itself will have to be questioned. With the present knowledge, however, we can proceed a little further.

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* This paper was published in Japanese in the SORYUSHI-RON KENKYU 1 (1949), 99.

According to the relativistic quantum theory, a particle having a rest mass and being in a stationary state cannot be confined in a domain with a diameter smaller than $\frac{\hbar}{mc}$. So long as the fundamental characteristics of the present relativistic quantum theory are kept unchanged, the Compton wave-length has a significant role to play in the theory of elementary particles. No matter to what extent it is carried over to the future theories, this length will at least have an important influence in the transition phase of the theories.

The following is the summary of significant lengths in the present theory.

$\frac{\hbar}{mc}=137r_0$: The Compton wave-length of the electron,
$\frac{e^2}{mc^2}=r_0=2.80\times 10^{-13}\text{cm}$: The classical electron radius,
$\frac{\hbar}{m_\pi c}=\frac{137}{280}r_0\sim\frac{1}{2}r_0$: The range of the nuclear force,
$\frac{\hbar}{m_c c}=\frac{137}{1,000}r_0\sim\frac{1}{7}r_0$: The Compton wave-length of the *C*-meson,
$\frac{\hbar}{Mc}=\frac{137}{1,800}r_0\sim\frac{1}{13}r_0$: The Compton wave-length of the nucleon,
$\frac{1}{10}r_0$: The cut off range of Coulomb field from the *C*-meson· theory of the mirror nuclei,
$\frac{e^2}{Mc^2}=\frac{1}{1,800}r_0$: The classical nucleon radius.

Let us consider the possibility that the nucleon should consist of smaller elements. If the nucleon consists of particles of as small a mass as μ-meson (Klein[1]), their individual dimension should be at least $\frac{\hbar}{m_\mu c}\sim r_0$. This will be an important restriction on the dimensions of elemetary particles.

Another important problem on the elementarity of the nucleon is to what extent each nucleon persists its individuality in the nucleus. For instance, is it possible to treat a deuteron or an α-particle as an elementary particle? In other words, can the photo-dissociation of the deuteron be treated with a field attributable to the deuteron as an elementary particle? This possibility, however, is refused by A. Bohr[2]. Taking account of Nafe, and Nelson's[3] measurement of the hyperfine structure of deuterium, A. Bohr concluded that the electron rotates around the proton within the *D*-nucleus and is independent of the magnetic moment of the neutron. In other words, the proton is known to have the individuality in the *D*-nucleus.

The method taken by A. Bohr can also be applied to the inner structure of the proton. Thus we shall have an interesting problem on the anomalous magnetic moment of the proton. If the proton is supposed to be an aggregate of such particles as μ-mesons which obey the Dirac equation, we should expect still larger anomalous magnetic moment.

Finally, we have the problem of the mirror nucleus as an important scaffolding. Wightman[4)] derived from the mass defect of the mirror nucleus a conclusion that Coulomb field prevails up to the points $\frac{1}{10}r_0$ distant from the center of the proton. His calculation is subject to criticism in some respects, but if it be admitted, the radius of the proton must be smaller than $\frac{1}{10}r_0$, and it is unlikely for the proton to be composed of smaller particles, since the Compton wave-length of the proton is $\frac{1}{13}r_0$.

If the nucleon consists of the particles as small as π-mesons, the dimension of the nucleon is to be considered as about $\frac{1}{7}r_0$.

2. On the *C*-meson.

The *C*-meson was introduced in the theory to play an important role in problems relating to the structure of elementary particles. At a meeting of the Physical Society of Japan held in 1948, Sakata, Taketani and Inoue set forth an opinion on its observability, to the effect that the influence of the *C*-meson is appreciable only in such a case that the domain $\sim \frac{\hbar}{m_c c}$ comes into question. In the case of electrons, the influence of *C*-meson will not appear because $m \ll m_c$ for electrons, and the force range and the elementary length have no significance accordingly. This indicates that the *C*-meson theory is in agreement with the Tomonaga-Schwinger's theory in their results, and that the latter can be considered as the extreme case where $m_c \rightarrow \infty$.

The above conclusion of ours has been verified by various calculations.[5)] As for the μ-meson, if $m_\mu \sim m_c$, we can expect to some extent the observability of *C*-mesons. Further, for the proton, considerable influence of the *C*-meson will appear. However, it is still questionable how clearly we can observe the *C*-meson, because the behavior of the nuclear forces is very complicated. The recent observation of mass difference between the charged and neutral π-mesons will provide a check for *C*-meson. But no definite results can be derived from it, because the difficulties and ambiguities of the present field theory is more acute in the case of mesons than in the case of Dirac electrons.

At the beginning of 1949 Taketani[6)] estimated the observability of the *C*-meson and reported that in the bremsung of those heavy particles *C*-meson's influences will become important especially in view of larger energy flow to soft components through C-meson bremsung.

That is to say, in the bremsung of heavy particles the emission probability of C-mesons is larger than that of γ-rays, and the emitted C-meson is immediately converted into an electron-positron pair. Nagoya group is advancing the calculation on this line. On the other hand, Hayakawa, Fujimoto and Yamaguchi[7] have investigated the similar processes for the μ-meson. In consequence m_c was found to be larger than what we had expected before, and to have a mass over $500\,m$, which seems to be in agreement with other theoretical results. Further, it seems to have some connection with the τ-mesons or varitrons.

3. On the nuclear force.

In advancing the research on the nuclear force, we are liable to get into confusions unless we take a careful methodological procedure; an example can be seen in Møller-Rosenfeld's treatment, in which they attempted to determine the type of meson field from the elimination condition for the singularity of $\frac{1}{r^3}$.

The behavior in the nearer neighborhood of the nucleon involves many problems: the problem of strong coupling, the problem of relativistic formulation.

As there are many problems to be discussed in relation with the nearer neighborhood of the nucleus, like the problem of strong coupling, relativistic formulation, and the C-meson in the nuclear field, it is not certain how much significance the elimination of ∞ by the mixed field will have.

The method we have taken (Taketani, Nakamura, Sasaki[8]), therefore, was to separate the problems at distance from the nucleon from those in the neighborhood. In order to determine the type of meson field we utilized the phenomena at a distance from the nucleon (those just on or farther than the limit of the force range), and excluded the phenomena in the nucleon's neighborhood by the cut-off method.

The phenomena at distance like the problem of deuteron, including its quadrupole moment and P-N and P-P scattering at $E=100$ MeV, can serve to a considerable extent to determine the type of field. Besides the above, the emission, absorption and scattering of the π-meson, π-μ decay, absorption of the μ-meson by nucleus, and γ-π transition, may also serve to the same purpose, which are not disturbed by the above mentioned difficulties appearing within the range when the energy is small.

That is to say, it is necessary for the research in the meson theory to separate the problems within the force range from those without for the present purpose. This is a powerful method to be purposely taken in the substantialistic stage of the elementary particle theory.

4. On the very strong nuclear forces.

Fermi[9] proposed a theory according to which π mesons are not elementary particles, but are composed of nucleons bound by very strong nuclear forces. By these binding almost all of the mass of nucleons annihilates and π-mesons are built up from the particles with much heavier masses.

But under such strong interactions, it is quite questionable, whether the concept of the elementary particle or the composite particle has any meaning. And Fermi's treatment by perturbation calculation seems to be meaningless, as has been already pointed out by himself.

Further if we accept Fermi's theory, the discrepancy will appear in the competition between (π-μ) and (π-β) decays.

Here all the interactions between π-mesons and other elementary particles are interpreted to originate through nucleons. In the case of β-decay of a nucleus, therefore, there are two ways of processes, that is, the process of the direct emission from the nuclear particles and that through π-meson. It is considered, however, that the latter is rather favorable, because of the strong binding of nucleons in π-mesons, and thus the interaction scheme of the elementary processes has the similarity of the Sakata's original theory. Consequently, if we adjust the coupling constants for β-decay and μ-meson capture to the experimental values, the (π-β) decays will be faster than the (π-μ) decay which contradicts the experimental evidences.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Remarks on the Field Theory

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(Received August 8, 1950)

1. Physical interpretation of the field theory.

The method of quantization of field gives us a formal theory of interacting fields. Although the complete development of the theory was impeded by divergence difficulties which were inevitably accompanied with it, it provided us with a consistent method of describing interacting fields. The quantization of field was carried out in conformity with the principle of correspondence; the classical field remained in the quantized theory formally as it was and the particle aspect of theory was manifested by transforming the field quantity into the configuration space or into the momentum space. However, this correspondence between the field and the particle seems too formal. The correspondence is indeed very clear when there is no interaction among the fields and the commutation relations express the measurability of field quantities. But the situation is not so clear when there are interactions.

The only successful method of treating the interacting fields we have had hitherto is that of perturbation and this method so well matched to our mode of thinking that it has become almost impossible to think in any other manner than that of free particles and their transitions through virtual processes caused by the interactions. Strictly speaking, this way of treatment would have physical meanings if there were no interaction when the system under consideration was prepared and then the interactions were switched in during some time interval and the observation were made after it being switched off. But such ideal cases are seldom encountered in reality. If we consider, e. g., the Compton effect, an electron and a photon of definite momenta and energies are brought to collision and, afterwards, scattered electrons and photons are observed. In this case, what we mean by the free electron or the free photon is not a "bare" particle, but it is a particle constantly interacting with its surroundings. Such a "true" particle was treated in a nonrelativistic manner first by Bloch and Nordsieck(1) in connection with the problem of bremsstrahlung, and more extensively discussed by many authors(2) from the new stand point of the amalgamation theory of Tomonaga and Schwinger. It was shown by them that all divergences could be removed simply by altering the magnitudes of masses and charges of the particles. Although the magnitudes to be renormalized were infinite in the present theory, definite results thus became possible to be obtained for any physically significant quantities.

The most elaborate investigation of Dyson(3) has shown that the success of the amalgamation theory is not restricted by the perturbation method as long as we deal with processes which are described by means of S-matrix.

In spite of this success many problems are still remaining unsolved. One of the most important and difficult problems among them will be to establish a field theory which involves no divergences from the very first. Indeed, if all divergences are removed by the amalgamation, all the matrix elements of the S-matrix become finite; but this does not mean we can easily construct a divergence-free field theory without making a round through the amalgamation process. Moreover, it has not yet been proved if we can always obtain finite results in more general cases which require more general description than that of the S-matrix. In order we may answer it, we will have to solve such questions as follows: What will be the charge of the electron immediately after it is created in a meson decay? and is there any reason for the fact that all elementary particles have charges integral multiple of the elementary charge in spite of the situation that the charge renormalization will in general different for different types of particle? and especially why the neutron has no charge at all?

Generally speaking, it will be necessary to know what state is prepared by a given arrangement of apparatuses and how the state is expressed as a function of the field variables.

2. Definition of the vacuum and free particles.

In the first place, we consider relatively simple problems. Let the electron field be described in the Heisenberg representation by ψ and its adjoint $\bar{\psi}$ and the electromagnetic field by $A_\mu\,(\mu=1,2,3,4)$. If there were no interaction between them, the definition of an negaton, a positon or a photon would be very simple and the vacuum would be defined as a state where there is no such particles. As is well known, ψ satisfy the Dirac equation in the free space and its positive-frequency part is interpreted as the operator that decreases the number of negaton. Similar procedure is also applicable to the photon. The situation becomes very complicated when an interaction is introduced among them. In the first place, ψ does no more satisfy the homogeneous Dirac equation and consequently it is no more a superposition of free electrons. It is usual to expand ψ into the complete set of free-electron eigenfunction and to interpret its Fourier coefficients belonging to positive and negative frequency eigenfunctions as annihilating operators of negaton and creating operators of positon respectively. But this procedure is clearly not gauge invariant. In order that the theory is gauge invariant, all observable quantities must be invariant under the gauge transformation:

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu G(x) \quad (\partial_\mu = \partial/\partial x_\mu),$$

$$\psi(x) \rightarrow e^{-ieG(x)}\,\psi(x), \tag{1}$$

where $G(x)$ is an arbitrary function satisfying the wave equation $\Box^2 G(x)=0$. We see thus that the above-mentioned definition of annihilating or creating operators is not gauge invariant.

Another possibility is the procedure of Schwinger. As in the interaction representation ψ and A_μ satisfy the free-field wave equations they seem to be more suitable to define the particles. But the situation is not so simple, because any measurable quantity must be invariant also in this case under the same gauge transformation (1), as proved by Koba, Tati and Tomonaga.(4) Similar procedure is suggested by Yang and Feldman.(5) In the Heisenberg representation, they introduce the incoming waves $\mathring{\psi}$ and $\mathring{A}_\mu$ by

$$A_\mu(x)=\mathring{A}_\mu(x)-\int_{-\infty}^{x} D(x-x')\, j_\mu(x')\, d^4x',$$

$$\psi(x)=\mathring{\psi}(x)+i\int_{-\infty}^{x} e\, S(x-x')\gamma_\nu A_\nu(x')\psi(x')\, d^4x', \tag{2}$$

where j_μ, D and S have the ordinary meanings and the integration must be extended from a space-like surface through x to the past half-space. The incoming waves $\mathring{\psi}$ and $\mathring{A}_\mu$ satisfy the free-field wave equations and commutation relations. This suggests that we shall be able to define the annihilating operator of negaton by the positive-frequency part $\mathring{\psi}^{(+)}$ of $\mathring{\psi}$. But this expectation proves soon to be false because $\mathring{\psi}$ is not gauge invariant. The change of $\mathring{\psi}$ under the gauge transformation is not so simple as (1), but is calculated from its change at the infinite past as ψ must coincide with $\mathring{\psi}$ there. It will never be gauge invariant unless there had been no interaction in the infinite past.

From these considerations we see that the definitions of the vacuum and particles hitherto given are not gauge invariant. But the above considerations seem to suggest the postulate that the particle operators will be able to be defined by $\mathring{\psi}$ and $\mathring{A}_\mu$ of free fields which are connected with ψ and A_μ respectively through an imaginary process that $\mathring{\psi}$ and $\mathring{A}_\mu$ become ψ and A_μ when the charge e was initially zero in the infinite past and it was increased gradually to the present value. If this "adiabatic" correspondence of $\mathring{\psi}$ and $\mathring{A}_\mu$ to ψ and A_μ works or not must be examined in more detail.

3. The adiabatic postulate.

To formulate mathematically the above-mentioned correspondence seems very difficult. We shall tentatively assume that, even if the charge is not a constant but a slowly varying function in the space-time, ψ and A_μ satisfy the ordinary wave equations:

$$\{\gamma_\nu(\partial_\nu-ie(x)A_\nu(x))+m\}\psi(x)=0,$$

$$\square^2 A_\mu(x) = -j_\mu(x), \qquad (3)$$

$$j_\mu(x) = ie(x)\bar{\psi}(x)\gamma_\mu\psi(x),$$

and that the gauge transformation (1) still holds. Then $\mathring{\psi}$ will be invariant under the gauge transformation, because it approaches ψ in the infinite past and the charge is zero there. On the other hand, $\mathring{A}_\mu$ suffers the same gauge transformation as A_μ in (1). In other words, it may be expected that, if we start from different gauges, we shall obtain different ψ and A_μ which will be connected by the relation (1). If this is true, we can define the negaton or positon in a similar way as in the case of the bare electron field. Photons will be defined by $\mathring{A}_\mu$ as usual. Especially the vacuum will be defined by

$$\mathring{\psi}^{(+)}(x)\,\Phi_0 = 0, \quad \mathring{\bar{\psi}}^{(+)}(x)\,\Phi_0 = 0, \quad \mathring{a}_\mu^{(+)}(x)\,\Phi_0 = 0, \qquad (4)$$

where $\mathring{a}_\mu$ is the transverse part of $\mathring{A}_\mu$. A state in which only one electron exists will correspond to a state in which there is only one bare electron in the infinite past. Generally, a state in which there are several bare electrons and photons will correspond to the state of system that, when it was prepared, there were several true free electrons or photons whose wave packets were spatially far apart.

Strictly speaking, the gauge transformation (1) will not be consistent with the equations of motion (3), and the above expectation must be justified by a more rigorous formulation. But we shall show that this expectation is plausible. $\mathring{\psi}$ defined by (2) transforms under the gauge transformation as follows:

$$\mathring{\psi}(x) \to \psi(x)e^{-ie(x)G(x)} - \int_{-\infty}^{x} ie(x')S(x-x')\gamma_\nu(A_\nu(x') - \partial'_\nu G(x'))e^{-ie(x')G(x')}d^4x'.$$

Now, by using the equation (3) for ψ and $\bar{\psi}$ and the equation for $S(x)$, we have

$$ie(x')S(x-x')\gamma_\nu A_\nu(x')\psi(x') = S(x-x')\gamma_\nu\partial'_\nu\psi(x') + \partial'_\nu\gamma_\nu S(x-x')\cdot\psi(x'),$$

and the above expression for $\mathring{\psi}$ can be written as

$$\mathring{\psi}(x) \to \psi(x)e^{-ie(x)G(x)} - \int_{-\infty}^{x} \{\partial'_\nu(S(x-x')\gamma_\nu\psi(x')e^{-ie(x')G(x')}) -$$

$$- i\,S(x-x')\gamma_\nu G(x')\psi(x')e^{-ie(x')G(x')}\cdot\partial'_\nu e(x')\}\,d^4x'$$

$$= \int_{-\infty} S(x-x')\gamma_\nu\psi(x')\,d\sigma'_\nu + i\int_{-\infty}^{x} S(x-x')\gamma_\nu G(x')\psi(x')e^{-ie(x')G(x')}\partial'_\nu e(x')\,d^4x'.$$

If we assume that any such integral involving the derivation of e contributes nothing in the limit $|\partial_\mu e(x)/e(x)| \to 0$, $\mathring{\psi}$ will indeed be gauge invariant.

As an example of necessity of this assumption, let us consider the external vacuum polarization. The induced charge and current is calculated by taking vacuum expectation value of j_μ. As is well known, it is given by

$$<j_\mu(x)>_0 = i\int_{-\infty}^{x} <[\,j_\mu(x), j_\nu(x')\,]>_0 A_\nu(x')d^4x'$$

$$= 4e(x)\int_{-\infty}^{x} e(x')G_{\mu\nu}(x-x')A_\nu(x')d^4x', \qquad (5)$$

where

$$G_{\mu\nu}(x) = \partial_\mu\Delta\,\partial_\nu\Delta^{(1)} + \partial_\nu\Delta\,\partial_\mu\Delta^{(1)} - \delta_{\mu\nu}[\partial_\lambda\Delta\,\partial_\lambda\Delta^{(1)} + m^2\Delta\Delta^{(1)}] \qquad (6)$$

satisfy the continuity equation

$$\partial_\nu G_{\mu\nu}(x) = 0. \qquad (7)$$

The gauge invariance of (5) is proved just as we have done in the proof of invariance of $\hat{\psi}$. Thus, under the transformation (1), $<j_\mu(x)>_0$ is transformed into

$$<j_\mu(x)>_0 - 4\,e(x)\int_{-\infty}^{x} e(x')\partial'_\nu\{G_{\mu\nu}(x-x')G(x')\}d^4x'.$$

The last integral reduces to zero if the derivative of e can be neglected and e was zero in the infinite past.

According to Schwinger, $G_{\mu\nu}$ is simplified to

$$G_{\mu\nu}(x) = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box^2)G(x),$$

$$G(x) = -\frac{1}{24\pi^2}\lim_{P\to\infty}\left(\log\frac{P+\sqrt{P^2+m^2}}{m} - 1\right)\cdot\delta^4(x)$$

$$-\frac{1}{64\pi^2}\frac{\Box^2}{m}\left(F_1(x) - \frac{1}{3}F_2(x)\right). \qquad (8)$$

As is well known, the first term of $G(x)$ involving $\delta^4(x)$ gives rise to a current and charge proportional to the external one. This is nothing but the charge renormalization. But the integral in (5) is extended only over a past half-space, and the incomplete integration of the delta-function has no definite value. If the integration is stopped just before the region approaches the point x, the value of the integral will be zero. Thus we see that the infinite charge renormalization is not the necessary conclusion of the theory. What value be given to it, the conservation of charge is not destroyed. This problem must be re-examined from a wider point of view.

In this paper we remarked that there is still many problems to be solved even in the framework of the present theory. More detailed investigation will be made in another occasion.

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Progress of Theoretical Physics, Vol. 5, No. 4, July~August, 1950

Analysis on the Two Meson Theory

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(Received August 8, 1950)

Abstract

Possible models for the two mesons, π and μ are selected by testing them with the present experimental knowledges, concerning the β-decay, the μ-meson capture, and the π-μ decay.

1. Introduction.

After the discoveries of π and μ mesons, various models for these two mesons are discussed extensively by Yukawa,[1] Taketani,[2] Wheeler,[3] etc. Of these the choice of the models I, II, and III will be analysed in this paper.

2. β-Decay. (model I)

The model I proposed by Inoue and Ogawa[4] following Sakata-Tanikawa's model of the two mesons, is based upon the original Yukawa's idea; that is the beta-decay of the nucleus goes through virtual π mesons. It is generally believed that this model has a defect that a free π meson decays faster into the light particles than into μ mesons. So far as the scalar, vector and pseudovector π mesons are concerned, there is no room for doubt. For pseudoscalar π meson, as was once pointed out by Ozaki[5] and Shono[6] partly, this defect can be removed, if the interactions between $(\pi-e, \nu)$ and $(\pi-\mu, \nu)$ could be limited to the pseudovector type. But in this case, it is claimed that usual perturbation calculation cancels out the matrix elements of the allowed transitions of β-decay through π mesons, and it starts from the matrix elements of $\int \gamma_5$.

If it is true, the troubles accompanying this result are that the Sargent law of the allowed β-decay and the forbidden beta spectra can not consistently be reproduced. Meanwhile, the choice of the direct interaction in the original Hamiltonian may be adjusted to bring about the allowed type transition, as it was proposed by Miyazima[7] and Sakata,[8] in which case the model of Inoue and Ogawa can escape the difficulty of the Sargent law. Of course, in this case the β-decay of the nucleus is to be explained by mixing of meson theory and Fermi theory. In this point, Lagrangian method recently proposed by Koba[9] will settle the ambiguities concerning direct interactions involved in the Hamiltonian. But if the experiment now going on the forbidden β-ray spectra requires the interaction other than that of pseudovector type,[10] this will not hold good. Anyhow, meson theory of β-decay will not conclusively be excluded for the time being.

3. π-μ Decay of Spin $\frac{1}{2}\hbar$ μ meson. (Model II)

The main problem on the model (II) is which process, $(\pi\rightarrow\mu, \nu)$ or $(\pi\rightarrow e, \nu)$, is more predominent if both takes place through virtual nucleon pair. So far various authors[11],[12],[13], have treated the process (II). According to Steinberger,[13]

(A) Only if π is pseudoscalar, we have the term $(m_1+m_2)^2$ (where m_1 and m_2 are the masses of two particles in the final state) in the matrix element as well as in the volume element of the momentum space for the final state. This term is 4×10^4 times smaller for $(\pi\rightarrow e, \nu)$ than for $(\pi\rightarrow\mu, \nu)$, which favors the agreement of the theory with the experiment.

(B) If one disposes of the divergent integral by regulator, the absolute value of life time for $(\pi\rightarrow\mu, \nu)$ becomes extraordinarily greater than the experimental value. Therefore, one cannot take this procedure.

On the other hand, according to Finkelstein and Ruderman,[12] the absolute life time of $(\pi\rightarrow\mu, \nu)$ can be adjusted to the experimental value only in the case of pseudoscalar π meson, by cutting off the logarithmic divergent term instead of using the regulator.

Table I. Selection rules for π-μ decay. (Spin $\frac{1}{2}\hbar$ μ meson)

π	π-N \ N-e, ν or N-μ, ν	S	V	t	PV	PS
S	S		*	*	L	L
	V	*	D	D	*	*
V	V	*			*	*
	T	*			*	*
PV	PV	L	*	*		ε
	T	*	ε		*	*
PS	PS	L	*	*		
	PV	L	*	*		

where * : Furry's Theorem
L : Lorentz Invariance
ε : Parity
D : Divergence Theorem
} prohibit the process.

Thus, in order to finish the analysis of this problem on the present stage, we shall check one by one the life times $\tau_{\pi\mu}$ of all kinds of π meson for all types of the interactions between nucleons and (μ, ν) or (e, ν).

As for the method of calculation, we followed the theory of Tomonaga and Schwinger.

1) Interaction.

The Hamiltonian for the direct coupling of light particles and nucleons is assumed to be the same form as that of μ meson and nucleons. As a result of the calculation following Tomonaga-Schwinger's method, we have obtained the following forms (2). The quadratically divergent term is caused by the same origin as in the case of the self-energy of the photon, which contradicts with the divergent theorem, the equivalent theorem and gauge invariance. For these reasons, it can be reasonaly dropped off.

2) Selection rules.

Taking into consideration various types of Fermi coupling between the nucleon, N, and (e,ν) or N and (μ,ν) and various types of mesons, we have the following selection rules. (See Table I)

This selection rule is exactly the same with that of Finkelstein and Ruderman but slightly differs from that of Steinberger.

3) The life times, $\tau_{\pi\mu}$, of π–μ decay are given as follows:

$$\frac{1}{\tau_{\pi\mu}}=\left(\frac{\mu}{2\pi}\right)^5\frac{1}{8}\sqrt{(1-\theta^2)(1-\theta'^2)}\,L, \tag{1}$$

where L takes the following forms depending on the type of π meson.

$$\left.\begin{aligned}
L_S&=18(1-\theta^2)(\lambda^{-2}A-B)^2 f_S^2 F_S^2,\\
L_V&=\frac{2}{3}(1-\theta'^2)[(2+\theta^2)(2BF_V+\lambda^{-1}AG_T)^2 f_V^2\\
&\quad+12\theta(2BF_V+\lambda^{-1}AG_T)\{\lambda^{-1}AF_V+(\lambda^{-2}A+B)G_T\}f_V f_T\\
&\quad+4(1+2\theta^2)\{\lambda^{-1}AF_V+(\lambda^{-2}A+B)G_T\}^2 f_T^2,\\
L_{PV}&=\frac{8}{3}(1-\theta^2)\{(2+\theta'^2)F_{PV}^2 f_{PV}^2+(1+2\theta'^2)G_T^2 f_T^2\}(\lambda^{-2}A-B)^2,\\
L_{PS}&=2(1-\theta'^2)\{f_{PS}[F_{PS}(-\lambda^{-2}A+3B)+G_{PV}\lambda^{-1}A]\\
&\qquad\qquad+f_{PV}\theta A\lambda^{-1}(-F_{PS}+2\lambda^{-1}G_{PV})\},\\
\theta&=\frac{m+m'}{\mu},\quad \theta'=\frac{m-m'}{\mu},\quad \lambda=\frac{\mu}{\kappa},\\
&m, m'\text{: (masses of }\mu\text{ meson and neutrino)}\times c,\\
&\kappa\text{: nucleon mass}\times c,\\
&\mu\text{: }\pi\text{ meson mass}\times c,
\end{aligned}\right\} \tag{2}$$

$$\begin{aligned}
A&=2\log\infty-\int_{-1}^{1}\frac{dv}{2}\log\left(1-\frac{1-v^2}{4}\lambda^2\right)\\
&=2\log\infty+\frac{1}{6}\lambda^2+\frac{1}{10}\lambda^4+\cdots,
\end{aligned} \tag{3}$$

$$\begin{aligned}
B&=\frac{1}{3}\log\infty-\int_{-1}^{1}\frac{dv}{2}\,\frac{1-v^2}{4}\log\left(1-\frac{1-v^2}{4}\lambda^2\right)\\
&=\frac{1}{3}\log\infty+\frac{1}{30}\lambda^2+\frac{1}{280}\lambda^4+\cdots,
\end{aligned} \tag{4}$$

$$\log\infty=\lim_{k\to\infty}\left(\log\left(\frac{k+\sqrt{k^2+\kappa^2}}{\kappa}\right)-\frac{k}{\sqrt{\kappa^2+k^2}}\right). \tag{5}$$

The value of $\log\infty$ cut off at suitably chosen value of k is denoted by $\bar{A}$.

Then the values of $\bar{A}$ and cut-off momentum are shown in the following Table II.

It is obvious from this table that the value of $\bar{A}$ is sensitively dependent upon the cut-off momentum.

Table II.

Cut off of k	κ	10κ	100κ
$\bar{A}$	0.18	1.77	4.1

4) The life of $\pi\to(e,\nu)$, $\tau_{\pi e}$.

If m and m' in the above $\tau_{\pi\mu}$ formula is substituted by the masses of the electron and the neutrino and Fermi coupling term of N and (μ,ν) by that of N and (e,ν), the resulting formula gives the values for $\tau_{\pi e}$.

Table III. $\tau_{\pi\mu}$ and $\tau_{\pi\mu}/\tau_{\pi e}$

spin $\frac{1}{2}$* μ meson

π	$N\longleftrightarrow\pi$	Fermi Coupling	$\tau_{\pi\mu}$ (see)			$\tau_{\pi\mu}/\tau_{\pi e}$
			Regulator	$\log\infty=0$	$\log\infty=\bar{A}$	
S	S	f_S	6.27	3.26×10^{-5}	$1.13\times10^{-10}A^{-2}$	5.7
V	V	f_V	4.26	4.26	$2.13\times10^{-5}A^{-2}$	4.4
	T	f_V	1.49×10^{-2}	1.49×10^{-2}	$5.03\times10^{-8}A^{-2}$	4.4
	V	f_T	1.06×10^{-2}	1.06×10^{-2}	$1.54\times10^{-8}A^{-2}$	2.9
	T	f_T	2.38	1.01×10^{-4}	$3.33\times10^{-10}A^{-2}$	2.7
PV	PV	f_{PV}	17.42	8.13×10^{-5}	$2.71\times10^{-2}A^{-2}$	4.4
	T	f_T	21.38	9.98×10^{-5}	$3.33\times10^{-10}A^{-2}$	2.7
PS	PS	f_{PS}	22.57	3.26×10^{-4}	$9.78\times10^{-10}A^{-2}$	5.7
	PV	f_{PS}	1.30×10^{-2}	1.30×10^{-2}	$4.51\times10^{-8}A^{-2}$	5.7
	PS	f_{PV}	2.23×10^{-2}	2.23×10^{-2}	$1.89\times10^{-3}A^{-2}$	1.3×10^{-4}
	PV	f_{PV}	$2.57\times10^{+1}$	1.26×10^{-4}	$4.06\times10^{-10}A^{-2}$	1.3×10^{-4}
Spin 0 μ meson						
S	S	f_S	1.2×10^{-2}	1.2×10^{-2}	$3\times10^{-8}A^{-2}$	1.2×10^{-3}

5) Results.

Substituting the numerical values into the above results of the calculation, we obtain the following Table III of $\tau_{\pi\mu}$ and $\tau_{\pi\mu}/\tau_{\pi e}$. These should be compared with the experimental values

$$\tau_{\pi\mu}\sim10^{-8}\text{ sec.},$$

$$\tau_{\pi\mu}/\tau_{\pi e}\leq10^{-2}.$$

where the constants are given as follows:

$$\tau_0=F^2f^2\left(\frac{\mu}{2\pi}\right)^5\simeq 5.63\times 10^{-7}, \tag{6}$$

$$\frac{F^2}{4\pi}=\frac{G^2}{4\pi}=0.1,$$

$2\pi^2\hbar^2/f^2m_e^5c^5=4.4\times 10^3$ sec : which corresponds the characteristic time of β-decay. This is adjusted in agreement with the value obtained by J. A. Wheeler and J. Tiomno[14] for the Fermi coupling constant of β-decay ;

$$f=2\times 10^{-49}\ \text{g}\cdot\text{cm}^3.$$

The characteristic time of β-decay is usually taken to be 3×10^3 sec. In this case the factor 30/44 must be multiplied to the life time of π-μ decay shown in the table.

From this calculation, we have the following conclusions.

A) Competition of $\pi\rightarrow(e, \nu)$ and $\pi\rightarrow(\mu, \nu)$.

Except the case of pseudoscalar π meson and pseudovector Fermi coupling, all the results are incompatible with the experiment. (c.f. $\tau_{\pi\mu}\ \tau_{\pi e}\geqq 10^{-2}$ in Table III)

B) The life time of π-μ decay.

Different results are deduced according to the treatment of the divergent integrals.

$$\left.\begin{array}{ll}\text{i)} & \text{Regulator}\\ \text{ii)} & \log\infty=\bar{A}\end{array}\right\}\text{give results which are at variances with the experiments.}$$

iii) cut off

With the suitable choice the cut-off momentum k, we can in general deduce the results which agree with the experiments. (c.f. $\log\infty=\bar{A}$ in Table III). Let us examine the case where the agreement with the experiment is exclusively attained for the competition of $(\pi\rightarrow\mu, \nu)$ and $(\pi\rightarrow e, \nu)$.

π ; Pseudoscalar

$$\left.\begin{array}{l}\text{Fermi couplings between } (N\leftrightarrow e, \nu) \text{ or } (N\leftrightarrow\mu, \nu)\\ \qquad\text{involves pseudovector.}\\ \text{If the coupling of } \pi\leftrightarrow N \text{ is pseudoscalar, } k=100\\ \qquad\text{and } \tau_{\pi\mu}=10^{-8} \text{ sec. and if that of } \pi\leftrightarrow N \text{ is}\\ \qquad\text{pseudovector, } k=\infty \text{ and } \tau_{\pi\mu}=1.25\times 10^{-8} \text{ sec.}\end{array}\right\} \tag{7}$$

Therefore, it is found out that, in the conclusion of Finkelstein et al, the case (7) was overlooked.

C) Selection Rule for β-decay.

In the recent analysis[10] of the forbidden spectrum of β-decay, it was concluded that the Fermi coupling must involve pseudoscalar and pseudovector as well as tensor interactions. According to the conclusion of the above section, however, the Fermi couplings can not involve pseudoscalar type in order that with the model II we may expect consistency with the empirical result $\tau_{\pi\mu}/\tau_{\pi e} \leq 10^{-2}$. Therefore, taking into consideration on both the two facts, we can conclude:

In the model II, π meson must be pseudoscalar and Fermi couplings must be pseudovector (for the latters, scalar, tensor, vector are available, but pseudoscalar must be excluded).

4. π-μ Decay of Spin 0 μ meson (model II)

μ meson of spin 0 was involved as one of the models in the two meson theory put forward by Tanikawa[15] in 1942. According to Tiomno,[16] the electronic spectrum due to the β-decay of μ meson can be reproduced by assuming suitable interactions between μ and the light particles. R. Latter and R. F. Christ[17] takes the same model (III) for the calculation of π-μ decay and μ-capture.

We shall now discuss the validity of assumption of spin 0 μ mesons in the model II. The method of analysis will follow the above procedures taken for the μ meson of spin $\frac{1}{2}\hbar$.

1) Probability of μ-capture (**Spin 0 μ meson**).

Capture life time for capturing μ meson in the K-shell is calculated.

$$\frac{1}{\tau_{\mu\ cap}} = \frac{|M|^2}{\tau_0} \frac{1}{4\pi} \frac{[4(1-\varepsilon^2)]^{\gamma+\frac{3}{2}}}{\Gamma(2\gamma+3)} \left(\frac{m_\mu (c r_0)}{\hbar}\right)^{2\gamma} \left(\frac{\Delta w + \varepsilon}{\varepsilon}\right), \tag{8}$$

where $\tau_0 = \dfrac{4\pi\hbar^5}{m_\mu{}^3 f^2 c^2}$, m_μ is the mass of μ meson. (9)

On the other hand, experimental evidences for μ capture is given, for $Z = 10$, as follows:

$$\tau_{\mu\ cap} = 1.49 \times 10^{-6} \text{ sec.}\ ^{18)}$$

If we put $\Delta w = 20/276$, we have from (8)

$$\tau_0 = 1.1 \times 10^{-7} \text{ sec.}$$

The direct coupling, f, of $(N \leftrightarrow \mu, \nu)$ derived from the experiment of μ-capture will be given

$$\frac{f^2 m_\mu{}^3 c^2}{\hbar^5} = 3.6\pi \times 10^7 \text{ sec}^{-1}. \tag{10}$$

2) π-μ decay (**Spin 0 μ meson**).

π-μ decay of π-mesons in the case of spin 0 μ mesons will be dealt with just the same way as in the case of spin $\frac{\hbar}{2}$ μ mesons. Selection rules for the processes will be given in Table IV.

According to the table, pseudoscalar π-meson must be eliminated since π-μ decay cannot take place at all. For the scalar π, the life $\tau_{\pi\mu}$ is given as follows:

$$1/\tau_{\pi\mu}=\frac{1}{8\pi}\left(1-\left(\frac{m_\mu}{m_\pi}\right)^2\right)\left(\frac{m_\pi}{2\pi}\right)^3\times$$
$$\times\frac{c^2f^2}{\hbar^5}\frac{F^2}{4\pi}[k]^2, \tag{11}$$

where $m_\mu/m_\pi=215/276$,

$$F^2/4\pi=0.1.$$

Table IV.
Selection rules for π-μ decay (Spin 0 μ meson)

π	S		V		PV		PS	
$\pi\longleftrightarrow N$ / $(N\longleftrightarrow\mu)$	S	V	V	T	T	PV	PV	PS
S		*	*	*	*	*	*	*
V	*	D				*	*	*
T	*	D				*	*	*
PV	*	*	*	*	*		E	*
PS	*	*	*	*	*		E	*

E: Equivalence

Inserting (10) into (11), we get

$$\tau_{\pi\mu}=3\times10^{-8}[\log\infty]^{-2}\text{ sec.} \tag{12}$$

log∞ is just the same meaning as before (see Table II).

As is obvious from this table, the regulator method and log ∞=0 procedure are at variances with experimental value of $\tau_{\pi\mu}$. If we cut off this log∞ at $k=10x$, $\tau_{\pi\mu}=10^{-8}$ sec. But the competition of π-μ decay and π-e decay cannot agree with the experiment. The analysis for the vector or pseudovector π-meson can be performed in just the same manner as before. In this case, too, the competition $\tau_{\pi\mu}/\tau_{\pi e}\gg1$, being contradictory with the experiment.

3) Results.

For the spin 0 μ, ν meson there remains the following possibilities for the type of π mesons consistent with experiment, provided the validity of cut off procedure, is assumed.

A) π: scalar and the Fermi couplings of $(N\leftrightarrow e,\nu)$ do not involve scalar interaction.

B) π: vector and the Fermi couplings of $(N\leftrightarrow e,\nu)$ do not involve vector and tensor interaction.

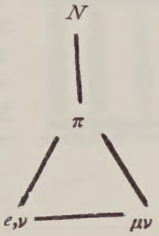
For the pseudoscalar π mesons, only if either of μ and ν mesons is scalar and pseudoscalar respectively, the selection rule can be satisfied. In this case the following condition is also required to explain the experiments concerned.

D) π is pseudoscalar and the Fermi couplings of $(N\leftrightarrow e,\nu)$ do not involve pseudoscalar interactions. Of course, the cut off procedure is assumed to be accepted.

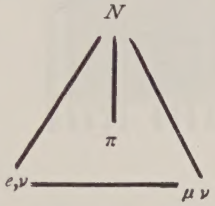
5. Disscussion of μ-capture (Model I, III)

For the model I, III some alteration is to be mentioned to our[2] previous discussion. We have mentioned before[2] that the nuclear matrix element for μ-meson capture $|\int\Psi\Phi|^2\leq1$. Since every proton in the initial nucleus has chance to capture the μ-meson, it should be $|\int \Psi \Phi|^2 \leq Z$, where Z is the atomic number of the nucleus. The important conclusion we can draw from this revision is that only the pseudoscalar π meson may be not at variances with the requirement of explaining μ-capture and π-μ decay by the model III and I. Another types for π meson, i.e. vector, pseudovector and scalar contradicts these experiments in explaining by the model III and I.

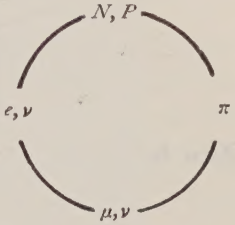
model I



model II



model III



There is no more question in so far as direct interaction between elementary particles is assumed. Indirect interactions through virtual process, such as β-decay and μ-capture for model I, π-μ decay for model II, and μ-capture for model III, are to be studied in the light of experimental evidences.

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